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MATHEMATICAL PROGRAMMING AND THE ANALYSIS OF CAPITAL BUDGETING PROBLEMS

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Associate Professor
of Finance
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of Technology

PRENTICE-HALL, INC.
Englewood Cliffs, N. J.

To My Parents

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Foreword

Professor Weingartner's dissertation, one of five selected for publication in the fourth annual Doctoral Dissertation Competition sponsored by the Program in Economic Development and Administration of The Ford Foundation, was completed during the academic year 1961-62.

The dissertation competition, which is open to doctoral candidates in business administration, the social sciences, and other fields relevant to the study of problems of business, is intended to recognize and encourage excellence in research on business by graduate students. In this year's competition the selection committee chose five such dissertations, to be added to the fifteen outstanding ones that have already been published. Publication awards are made to persons granted doctorates in business and related fields whose thesis research is especially distinguished by its analytical content and strong roots in the underlying disciplines common to business.

The dissertations selected this year in addition to Professor Weingartner's include the following:

The Effect of Capital Structure on the Cost of Capital
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James M. Ferguson
Department of Economics
University of Chicago

Statistical Decision and Related Techniques in Oil and Gas Exploration
Gordon M. Kaufman
Graduate School of Business Administration
Harvard University

On behalf of The Ford Foundation, I wish to express my gratitude to the members of the Editorial Committee for the care and thought they devoted to the selection process. The members of this Committee, who made the final selection of winning dissertations, are: Professor Alex Bavelas of Stanford University, Professor Roy Blough of Columbia University, and Professor Robert Ferber of the University of Illinois.

The Editorial Committee's task was considerably lightened by the assistance of ten readers, experts in the wide range of disciplines covered in the competition, who carefully screened the theses submitted. The Foundation joins the Committee in acknowledging their debt to Professors Thomas L. Whisler and Harry V. Roberts of the University of Chicago, Paul E. Breer of Cornell University, Lyman W. Porter and Earl F. Cheit of the University of California (Berkeley), Robert B. Fetter of Yale University, Daniel M. Holland of the Massachusetts Institute of Technology, Bernard Karsh of the University of Illinois, Donald P. Jacobs of Northwestern University, and Joseph W. Newman of Stanford University.

Finally, my colleagues and I wish to acknowledge the important contribution of Prentice-Hall, Inc. to the publication of the selected dissertations.

Clarence H. Faust
Vice President
The Ford Foundation

New York, N. Y.
January, 1963

Acknowledgements

This volume, which bears the authorship of one, in fact was the product of many people. The debt owed to them, and to those who contributed to its antecedents will not be extinguished by brief citations in a page or two.

I became aware of the subject of mathematical programming, and the field of operations research in general, only after coming to the Graduate School of Industrial Administration. My residency at the School was made possible by the grant of Mellon and Ford fellowships, and my debt to the School for its material support and intellectual stimulation is gratefully acknowledged.

I have benefited greatly from the environment that the School provided, and especially from my association with Professor Franco Modigliani, whom I served as research assistant, and Professors William W. Cooper and Herbert A. Simon, whose stimulation has led me to this undertaking.

This work would not have been possible without the unfailing help of my committee consisting of Professor Merton H. Miller, chairman, and Professors William W. Cooper and Franco Modigliani. They not only provided me with specific background for this dissertation, but, by insisting on and aiding me in my efforts toward greater rigor, forced me to develop and refine my ideas and to carry them forward to this written document.

The study was actually begun in 1959 and completed during my stay at the University of Chicago. My colleagues at the University, Professors Robert L. Graves and Lester G. Telser, also helped me to crystallize my thinking on these topics. Professor Graves must be singled out for his steadfast willingness to listen to my ideas as they were born, and for his innumerable comments and suggestions.

During the preparation of the dissertation for publication I have had the benefit of several discussions and correspondence with Dr. Ralph E. Gomory, upon whose work a part of Chapter 5 is based. His criticism of my original version has led to substantial revision of this section, and his cheerful help has been truly invaluable.

Special thanks are also due to Professor James H. Lorie who read that

part of the dissertation relating to his own work, and who helped to improve the style of the introduction. If the style of the rest of the dissertation is at all readable, it is only because of the persistent efforts of Professors Cooper and Miller, whose editorial contributions made it bearable for me to read what I had written.

This research also owes a debt to the Graduate School of Business of the University of Chicago for financial assistance which helped to defray the costs incidental to this work. The Operations Analysis Laboratory of the University of Chicago also made computer time available, as did the Argonne National Laboratory through the courtesy of the International Business Machines Corporation. My research assistant, Robert Lienhard, assisted in the preparation and execution of the computations. The final manuscript was cheerfully and speedily prepared by Harold Fibish and Ethel Jordan.

Executives of two large corporations also contributed to this work by explaining the nature of their capital expenditure planning in some detail, and by commenting on my ideas as they had been formulated.

A final acknowledgement is to my wife Joyce, who saw this dissertation through most of its many drafts. It would, indeed, be insufficient to thank her only for her typing. It was her continuing encouragement and support which has enabled me to carry this task to its completion.

Despite these acknowledgements for help gratefully received, it bears stating that I accept full responsibility for the resulting volume, including especially those errors which remain.

H. Martin Weingartner

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MATHEMATICAL PROGRAMMING AND THE ANALYSIS OF CAPITAL BUDGETING PROBLEMS

Chapter 1

Introduction

In recent years few topics in managerial economics have received as much attention as capital budgeting. Widespread interest in the subject is attested to by continuous outpouring of articles in the popular as well as the technical journals and in the steady increase in the number of books on the subject for both academic and business audiences.[1] This flow of material on investment decisions includes writings of numerous authors on the economics of the choice of public investments, especially in the development of water resources in this country, and of electric power projects abroad, notably in France.

This increased concern with capital budgeting comes at a time when considerable progress is being made in the application of mathematical methods and computers to the solution of complex business problems, but this progress has so far had relatively little impact on the main stream of work in capital budgeting.

This study is primarily an attempt to bring to bear linear programming and integer programming on certain long-standing problems of capital budgeting (specifically investment planning under capital rationing and under imperfect capital markets) that have resisted solution by the standard approaches in the field. In undertaking such a task, the objective is not so much to prescribe formulas or routine procedures for immediate adoption by business firms, but rather to develop a systematic approach to this class of budgeting problems to pave the way for eventual application to concrete problems. Since economists as well as business

[1] A goodly number of references to this voluminous literature will be found throughout the text and in the appended bibliography.

managers are concerned with capital budgeting, we shall also try, insofar as possible, to show the relation between the various programming formulations presented and the standard constructs of economic theory.

1.1. Background

Although discussion of capital theory in economics prior to World War II has focussed primarily on such ancient and fundamental issues as the nature and functions of the rate of interest, there was implicit in much of this discussion — and explicit in the writings of the more mathematical economist[2] — a model of optimal investment by the firm. Like most of economic theory at the time, this model made use of the simplifying assumptions of continuity and differentiability, certainty, and perfect capital markets. The key proposition of the model was that under an optimal investment program, investment would be carried by the firm to the point at which the marginal internal rate of return[3] was equal to the market rate of interest.[4]

This theory and the assumptions underlying it were entirely adequate for the very limited class of problems with which economists were then primarily concerned. In fact, crude as the model was, it did at least handle the element of time, whereas much of the management literature on investment planning — and much actual business practice — either neglected the time element entirely[5] or treated it in an even more grossly simplified way.[6]

If business practice has improved in this respect in recent years, much credit must be given to Joel Dean, whose writings brought home to so

---

[2]See K. Wicksell, *Lectures on Political Economy*, Eng. trans., E. Classen (New York: Macmillan Co., 1934); Irving Fisher, *The Theory of Interest* (New York: Macmillan Co., 1930); F. H. Knight, "The Quantity of Capital and the Rate of Interest," *Journal of Political Economy*, Nos. 4, 5 (1936).

[3]The internal rate of return of an investment project may be defined as that rate of interest for which the present values of all streams of costs and revenues associated with the project sum to zero.

[4]This and related propositions are discussed in a number of sources, two of which may be singled out for their noteworthy clarity: R. G. D. Allen, *Mathematical Analysis for Economists* (London: Macmillan Co., 1938), pp. 374-78; K. E. Boulding, *Economic Analysis*, 3d ed. (New York: Harper and Bros., 1955), Chapter 39.

[5]As, for example, in the use of the ratio of total revenue to total outlay to rank investments.

[6]For example, the "pay-back period" exemplifies a use of the time dimension for descriptive purposes and not for the evaluation of cash flows at different future dates.

many accountants and business managers the deficiencies in their capital budgeting procedures and provided them with a workable alternative approach based on the economic theory of investment.[7] Reduced to simplest terms, Dean's basic procedure for capital budgeting calls on the firm to compute the internal rate of return for each investment project and to rank these projects in decreasing order according to this criterion. The intersection of the resulting schedule with the marginal cost of capital schedule determines a "cut-off" or minimum acceptable rate of return which can then be used for selecting projects.

Although Dean's approach was an important step forward, the fact that it was a literal, if somewhat simplified, version of the economic theory of investment meant that it could guarantee an optimal result — in the usual economic sense of one that maximizes the economic welfare of the owners of the firm — only when the underlying assumptions of that theory were met, namely perfect certainty, perfect capital markets, continuous and continuously differentiable investment functions, and strict independence of investment projects. Unfortunately, one or more of these assumptions will frequently not be satisfied in actual business applications. If so, not only will the procedure not yield an optimum, but in some cases it is not even clear how Dean's prescribed procedure could be implemented at all.

These defects in the original Dean proposals have been the subject of much of the subsequent literature on capital budgeting. Particularly influential has been the well-known article of J. H. Lorie and L. J. Savage, "Three Problems in Rationing Capital,"[8] in which they showed clearly why and how Dean's rate of return criterion must fail whenever (a) the projects being evaluated are not independent; (b) the sum total of capital expenditures is constrained or limited in more than one time period; and (c) the stream of returns on a project contains alternations in sign. Lorie and Savage were able to develop reasonably satisfactory alternative procedures for handling cases (a) and (c). However, they were not particularly successful in dealing with the important case (b). Their procedure, which is of a trial and error nature, does not guarantee a solution within a finite number of steps, nor will it work when there are substantial indivisibilities or interrelations in the investment alternatives

---

[7] Joel Dean, *Capital Budgeting* (New York: Columbia University Press, 1951). Many of the concepts in this book are also presented in his *Managerial Economics* (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1951).

[8] *Journal of Business*, XXVIII, No. 4 (Oct., 1955), pp. 229-39; also reprinted in Ezra Solomon (ed.), *The Management of Corporate Capital* (Glencoe, Ill.: The Free Press, 1959).

being considered. It is these and related deficiencies in Lorie and Savage's attempted reformulation of Dean's proposals that provided the primary motivation for this work.

1.2. Plan of the Study

The first part of the volume will be devoted to an analysis of the Lorie and Savage problems and to their solution by use of mathematical programming methods. The problems are discussed in detail in Chapter 2, which also briefly reviews the methods of solution proposed by Lorie and Savage. In Chapter 3 the problems are formulated as linear programming models, and it is shown that this formulation permits the simultaneous resolution of a number of the difficulties encountered. In this context we also demonstrate that, although the programming model is only an approximation to the exact solution when indivisible investment alternatives are present, the approximation is a close one. A more exact approach, made possible by the recent development of integer programming, is applied to these problems in Chapter 4. The implications of the use of integer programming are analyzed in Chapter 5. In Chapter 6 the importance of fixed capital budgets and their use in business is discussed. Extensions of the linear programming model to cases involving multiple budgets, such as are required when limitations in raw material or executive personnel exist, are taken up in Chapter 7.

After having analyzed the Lorie-Savage problems we shall attempt to develop a more general approach to the problem of capital budgeting under capital rationing analogous to that presented by Charnes, Cooper, and Miller.[9] Their article opened a new avenue of approach, one which will be explored here in greater depth. In doing so our purpose will be in part to bring the computational power of linear programming to bear on these problems. In addition, we wish to utilize the richness provided by the duality relations of linear programming for clarifying and interpreting many aspects of capital budgeting which have not been wholly or effectively treated by existing methods, either in economic theory or in the literature of business budgeting. This formulation, which comprises the second part of this investigation, is undertaken in Chapter 8. It is exploited in Chapter 9 to analyze the implications for decision-making of various forms of capital market imperfections.

---

[9]A. Charnes, W. W. Cooper, and M. H. Miller, "Application of Linear Programming to Financial Budgeting and the Costing of Funds," *Journal of Business*, XXXII, No. 1 (Jan., 1959), pp. 20-46.

1.3. A Note on Uncertainty

The models proposed and examined all assume that future interest rates and future flows associated with investments are known with certainty. In making this assumption we follow the almost universal practice in the literature on capital budgeting. Justification for further building of investment models under certainty has been well stated by Trygve Haavelmo in his recent *A Study in the Theory of Investment.*[10] Some points raised by Haavelmo are directly applicable here. First, exact models may be used to advantage to gain insights. "It would be silly to deny the enormous practical importance that the intellectual play with exact theories has had."[11] One studies systems under "ideal" conditions, choosing to analyze the simple objects before attacking more complex ones. "For example, we might want to know what the result would be if a producer accurately maximized his profit under strictly given prices and an explicitly known technology. This may sometimes be no mean problem, even from a purely mathematical point of view."[12] "We are . . . far from having exhausted the amount of clarification and insight that can be gained from the study of exact models. We shall find more than enough to do even in a hypothetical world of non-stochastic models."[13] Although Haavelmo was concerned with macro-economic theories of investment, his comments apply with equal force to the field of capital budgeting.[14]

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[10]Chicago: The University of Chicago Press, 1960, pp. 17-21.

[11]*Ibid.*, p. 20.

[12]*Ibid.*

[13]*Ibid.*, p. 21.

[14]As the contents of *The Management of Corporate Capital* bear out, economists are far from unanimous on the implications of various measures used in evaluating capital expenditures and on the assumptions which underlie their use. The "popular" literature, such as the *Harvard Business Review*, also reflects this confounding when it publishes successive articles proposing one measure and criticizing the immediately prior one advanced, in almost endless variation. See, for example, J. G. McLean, "How to Evaluate New Capital Investments," XXXVI, No. 6 (Nov.-Dec., 1958); R. H. Baldwin, "How to Assess Investment Proposals," XXXVII, No. 3 (May-June, 1959); "From the Thoughtful Businessman," XXXVII, No. 5 (Sept.-Oct., 1959); as well as the editorial department, "Looking Around," XXXVI, No. 1 (Jan.-Feb., 1957). All these offer different solutions to the same problem.

CHAPTER 2

The Lorie-Savage Problem

We begin with a brief review of a capital budgeting problem first posed and partially solved by J. H. Lorie and L. J. Savage in their article, "Three Problems in Rationing Capital."[1] Their problem concerns a firm confronted with a variety of possible investment projects and a fixed capital budget. The cash flows associated with each project are given and the firm's cost of capital is assumed to be known and to be independent of the investment decisions. These assumptions make it possible to compute for each project a net present value defined as the algebraic sum of the elements of its stream of cash receipts and outlays discounted by the cost of capital. The objective is then to select from among the projects with positive present values[2] the particular projects which lead to the highest present value for the firm.

2.1. The Single-Period Case

In the special case in which the outlays on all projects take place in a single time period the solution proposed by Lorie and Savage[3] is to rank the projects by the ratios of their present value to cost[4] and then to select projects from the top of the list until the budget is exhausted. This

---

[1] *Op. cit.*

[2] We shall not distinguish between "net present value" and "present value" but simply use the latter term.

[3] *Op. cit.*, p. 231.

[4] That is, the "Profitability Index" in Robert N. Anthony, *Management Accounting*, Revised ed. (Homewood, Ill.: Richard D. Irwin, Inc., 1961) p. 554.

procedure does, in fact, maximize the present value of the firm under the indicated budget constraint, but only if there are no indivisibilities or further interrelationships to be considered — for example, interdependent projects.[5] As the authors point out, it is possible that a single, large project may have a high ratio of present value to cost, but its size may preclude the execution of a number of small projects which together would yield a higher net present value through fuller exploitation of the budget constraint. They comment, "Such discontinuities deserve special attention, but the general principles dealing with them will not be worked out here, primarily because we do not know them."[6] Their procedure fails when indivisibilities are present because their ranking is applied to projects singly, rather than to combinations of projects. The alternative of looking at all combinations of projects to select the *set* whose total present value is maximized subject to the same budget limitation completely changes the nature of the problem, and as we shall see, its solution requires the use of more powerful techniques.

2.2. The Multi-Period Case

The more interesting case is that in which the projects require outlays not in one, but in several time periods. Since specific budgets must be met in each of these future time periods as well as in the current one, the simple Lorie-Savage rule cannot be applied, of course, except when a budget ceiling is effective in only one year while funds are ample for all opportunities in other years.[7] For the general case, in which the budget constraint is binding in more than one period, Lorie and Savage do attempt to provide at least a solution in principle.[8] The logic of this solution can best be seen, perhaps, if we first restate the solution to the one-period problem in algebraic terms.

For the moment we will utilize the notation used by Lorie and

---

[5]The well-known criterion (see, for example, Dean, *Managerial Economics*, *op. cit.*, pp. 589 ff.) of ranking investment alternatives by their rates of return, then cutting off the list at the point at which the fixed funds are exhausted can easily be demonstrated to be nonoptimal, that is, inconsistent with maximization of the present value of the firm. (See also Lorie and Savage, *op. cit.*, p. 239.) Even the familiar rule of cutting off at the cost of capital may be shown to be incorrect when this cut-off rate is changing. The models of Chapter 9 will demonstrate this point.

[6]*Ibid.*, p. 232.

[7]An example in the appendix to this chapter shows that the presence of indivisibilities can create problems even in this situation.

[8]It will be shown in Chapter 4 that their method will not always work.

Savage, with certain slight modifications. Let y_j be the net present value of project j, that is, the discounted value of the stream of revenues generated by the project less the discounted value of the costs, where the latter as well as the former may occur in more than a single period, at least in the general case.[9] Let c_{tj} be the present value of outlays for project j in period t, that is, the amount of expenditure required for project j in period t discounted to the present. In the present case, $t = 1$. Also, let C_t be the present value of the expenditure ceiling in period t, that is, the amount of the ceiling discounted to the present. In this first instance, this may be denoted simply by C.

The one-period procedure consists of ranking the projects in decreasing order of y_j/c_{1j} and cutting off at $j = k$ just before

$$\sum_{j=1}^{k} c_{1j} < C \tag{2.1}$$

ceases to hold. At this cut-off point, for the chosen projects the quantity

$$y_j - p_1 c_{1j} \tag{2.2}$$

is positive or zero, and for the omitted projects this quantity is negative, where

$$p_1 \equiv \frac{y_k}{c_{1k}} \tag{2.3}$$

for the marginal project k.

Note that identity (2.3) defines the quantity p_1 which is required to determine the acceptable projects $j = 1, \ldots, k$ in the criterion (2.2). That is, the projects so selected involve present expenditure values whose total does not violate the budget constraint (2.1). The proper value of p_1 must be ascertained essentially by trial and error. Some positive trial value of p_1 is used to select the projects for which expression (2.2) is nonnegative, and the associated total present value of these expenditures is tested to see whether

$$C - \sum_{j=1}^{k} c_{1j} \tag{2.4}$$

is positive or negative. If it is negative, the value of p_1 is increased; if positive, it is decreased. The procedure ends when the accepted projects

---

[9]The choice of the appropriate rate by which future flows are to be discounted is not our primary concern. As mentioned, Lorie and Savage take this to be the cost of capital, although how this concept is to be measured is not discussed by them, and will not be discussed by us.

come as close as possible to exhausting the budget C. Actually, any value of p_1 in the range

$$\frac{y_{k+1}}{c_{1,k+1}} < p_1 \leq \frac{y_k}{c_{1k}} \tag{2.5}$$

will serve to separate the accepted projects from the rejected. Note also that selection of projects is the same whether

$$\frac{y_j}{c_{1j}} \quad \text{or} \quad y_j - p_1 c_{1j} \tag{2.6}$$

is computed, although only the second form (that is, that of expression (2.2) with the appropriate value of p_1) extends to the multi-period case.

For the multi-period problem, in which the selection process must take into account the budget limitations for several time periods, the Lorie and Savage procedure[10] calls for the trial and error choice of positive parameters $p_1, \ldots, p_t$ such that the quantity

$$y_j - \sum_{t=1}^{T} p_t c_{tj} \tag{2.7}$$

is positive or zero for the chosen projects, negative for the remaining ones. T is the farthest time period for which a budget is fixed and for which expenditures must be considered.[11] The values for p must be chosen so

TABLE 2.1. THE TWO-PERIOD LORIE AND SAVAGE EXAMPLE

Investment[a]	Present Value of Investment (y_j)	Present Value of Outlay in Period 1 (c_{1j})	Present Value of Outlay in Period 2 (c_{2j})
1*	\$14	\$12	\$ 3
2	17	54	7
3*	17	6	6
4*	15	6	2
5	40	30	35
6*	12	6	6
7	14	48	4
8	10	36	3
9*	12	18	3
Present Value of Budget Ceilings: C_1 = \$50		C_2 = \$20	

[a]Accepted investments are indicated by an *.

that the budget constraint is not violated in any period and so that there is no budget surplus that could be utilized for additional projects (aside from indivisibilities which are again neglected).

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[10]*Ibid.*, pp. 233 ff.

[11]This definition is admittedly ambiguous. It will be clarified in a later section.

A two-period example was given by Lorie and Savage.[12] We shall also find it useful to illustrate the problems and methods to be developed in later chapters and so we present it here, in Table 2.1. The desired values of p_1 and p_2 that Lorie and Savage found are 0.33 and 1, respectively. The quantity

$$y_j - p_1 c_{1j} - p_2 c_{2j}$$

is positive for investments 1, 3, 4, 6 and 9, as indicated by the asterisks.[13] Moreover, these projects fit the budgetary ceilings in each period since, in total, they require an outlay of \$48 in the first period and \$20 in the second period. These values of p_1 and p_2 have the property that an increase in one combined with a compensating decrease in the other will not allow the inclusion or substitution of projects which increase the total present value of the accepted set without also exceeding one or both of the budget ceilings of \$50 and \$20 in periods one and two, respectively. Lorie and Savage point out here that "a graphical process can usually lead to the correct values" to this problem although "finding the correct values for p_1 and p_2 is sometimes not easy."[14] The task is many times more difficult when problems of more than two periods are considered, and even when it appears that a likely set of p's has been found it may be difficult, short of trial by exhaustion, to demonstrate that the set is actually optimal.

2.3. The Method When Projects Are Not Independent

So far the discussion has been concerned entirely with independent investment opportunities. When projects are related to each other these relations must be taken into account. To facilitate further reference to the various situations which can arise, we shall set out the definitions of four forms of such relations between projects. The first two of these are considered by Lorie and Savage explicitly, while the other two could also be handled by their procedure.

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[12] *Ibid.*, Table 1, p. 234.

[13] For example, for investment 1 we have

$$y_1 - p_1 c_{11} - p_2 c_{21} = \$14 - \frac{1}{3} \cdot \$12 - 1 \cdot \$3 = \$7 > 0$$

while for investment 3 we have

$$y_3 - p_1 c_{13} - p_2 c_{23} = \$17 - \frac{1}{3} \cdot \$6 - 1 \cdot \$6 = \$9 > 0$$

and so on.

[14] *Ibid.* See also Section 3.5, and Fig. 3.3.

Independent projects: "When the worth of individual investment proposals is not profoundly affected by the acceptance of others."[15]

Mutually exclusive projects: When "acceptance of one proposal in such a set renders all others in the same set clearly unacceptable — or even unthinkable."[16]

Contingent projects: When acceptance of one proposal is dependent on acceptance of one or more other proposals.

Compound projects: When contingent projects are combined with the projects on which they depend, so that the independent project and the compound project may be treated as mutually exclusive alternatives.

As an example of mutually exclusive proposals we may cite several types of fork lift trucks which differ in the pallet sizes that they handle. In contrast to this, a fork lift truck and an accounting machine will, in general, be considered independent alternatives. An example of a contingent proposal would be a special boom for a crane which would have no value unless the crane itself were acquired; another would be radio-communication equipment for a fleet of trucks and the trucks themselves. The contingency relation need not be, and in general is not, symmetric.

A compound alternative could be constructed by combining the crane and the special boom and considering the crane so equipped as an alternative competitive to the same crane without the optional equipment. Similarly, the radio-equipped trucks, as a compound project, may be arrayed next to the fleet of trucks without radios as mutually exclusive alternatives. Contingent sets of alternatives can always be constructed as mutually exclusive projects, although other and more efficient procedures may be available with the method of solution we shall propose.

The Lorie-Savage procedure outlined above requires some rewording when non-independent, as well as independent proposals, are considered. To arrive at a choice among sets of mutually exclusive and independent projects simultaneously, one regards each independent project as the sole member of a set of mutually exclusive projects. The more general procedure then calls for choosing trial values of p_t and tentatively selecting from within each set of mutually exclusive projects that one for which

$$y_j - \sum_j p_t c_{tj} \tag{2.8}$$

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[15] *Ibid.*, p. 229.
[16] *Ibid.*

is maximum, provided that this quantity is also nonnegative.[17] If the latter condition does not obtain for some set of mutually exclusive projects, then no project is chosen from among the members (or combinations of members) of this set.[18] The list of proposals so obtained is tested by adding the expenditures required for the tentatively selected projects in each time period and comparing the totals with the corresponding budget ceiling. If a budget for some period is exceeded, the value of p associated with it is increased. If surplus funds remain, the value of p is decreased. In this way the values of p are modified and the list of projects is altered until an optimal set of investments is found.[19]

Summarizing, then, we may say that while Lorie and Savage have posed an interesting problem in capital budgeting, they have made little real progress toward its solution. The methods they propose do not deal adequately, and in all generality and rigor, with the very real problem of indivisibilities even in the simple, one-period case. The multi-period case is comprehended, at least in some conceptual sense, by the Lorie and Savage formulation, provided that indivisibilities are not present. However, even here the procedures they suggest have not been — and possibly cannot be — extended in a way that will provide the computational efficiency that is needed for actual applications.

It is to a development of possible alternatives that we shall now turn in order to provide a more promising route of development.

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[17]And provided that such designations do not violate the budgetary constraints, when cumulated.

[18]Rejecting all projects from among a set of mutually exclusive proposals is logically equivalent to accepting the alternative of "doing nothing," with present value of zero and required outlays of zero in each period, as Lorie and Savage point out (*ibid.*, p. 235).

[19]Lorie and Savage do not consider the problem of several collections of "equally best" projects.

APPENDIX 2A

A Special One-Period Case

A special case discussed by Lorie and Savage may be analyzed in some detail to advantage, since it sheds some light on a common problem in relation to governmental budgeting, in which only the forthcoming year's budgets are announced. The situation may best be described by quoting Lorie and Savage directly.

> Suppose that the opportunities available require more funds from either time period one or two than are permitted by the imposed constraints . . . . It is still relatively simple if (a) the best use of money available in period one does not exhaust the money available in period two or (b) the best use of money available in period two does not exhaust the money available in period one. In either case the optimum solution — that is, the solution which results in the greatest increment in the net worth of the firm, subject to the two stated constraints — is the one that makes the best possible use of the funds available for investment in one of the two time periods.[1]

Even for this case, we shall show that, contrary to their conjecture, the simple rule proposed cannot always be relied on to give a correct result. As in their treatment, we need concern ourselves only with the present values of the outlays and budgets and, of course, the net present values of the projects themselves.

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[1]*Op. cit.*, pp. 232-33.

Suppose, for example, that the choice is between two investment opportunities with present values and outlays as presented in Table 2A.1. Referring to the heading "Project 1" in this table, we see that a benefit can be secured from

TABLE 2A.1. NUMERICAL EXAMPLE OF THE SPECIAL ONE-PERIOD CASE

Row No.	Description	Project 1[a]	Project 2[a]
1	Net present value	\$30.00	\$28.00
2	Present value of outlay in year 1	\$ 6.00	\$18.00
3	Present value of outlay in year 2	\$19.00	\$ 3.00
4	Ratio of present value to present value of outlay in second year	$\frac{\$30}{\$19} = 1.58$	$\frac{\$28}{\$3} = 9.33^*$
5	Ratio of present value to sum of flows in both years	$\frac{\$30}{(\$19 + \$6)} = 1.20$	$\frac{\$28}{(\$3 + \$18)} = 1.33^*$
6	Ratio of present value to present value of outlay in first year	$\frac{\$30}{\$6} = 5.00^*$	$\frac{\$28}{\$18} = 1.56$

[a]Larger of the two ratios in the given row are indicated by an asterisk.

this project, whose net present value is \$30 (row 1). To obtain this benefit, however, requires expenditures in each of two years whose present values amount to \$6.00 in year 1 (see row 2) and \$19.00 in year 2 (see row 3), for a total present outlay value of \$25.00. Similar data are provided for "Project 2," whose row 1 entry shows a net present value of \$28.00, and whose total outlays, in present value terms, amount to \$21.00 — \$18.00 being incurred in the first year and \$3.00 in the second. We assume that the indicated projects represent a rigid schedule of commitments: an investment in either project must be decided upon as of now, otherwise the opportunity is completely foreclosed. Also, the resulting commitments to the indicated expenditures are binding so that there is no issue of undertaking one project now and eliminating, say, the year 2 expenditures by disposing of this project in favor of another competitor.

With these points in mind we now turn to a discussion of the problem at issue with reference to rows 4, 5 and 6 of this table. Consider first a budget for the second period with present value of \$20 and ample funds for both projects in the first period. Using the Lorie-Savage procedure for a one-period problem, as they suggest,[2] we compute the ratio of the present value to the discounted value of the required outlay in the second year for each project. This is shown on row 4 of the table. According to their prescription, the projects would be chosen in the order of these ratios, the highest being chosen first, until the \$20 is exhausted. In our example the second project would be preferred by this

[2]*Ibid.*

criterion. (The higher ratio, that in row 4, last column in the table has been marked by an asterisk.) The $3 cost of project 2 does not also permit acceptance of project 1 because an additional $19 would be required, and the budget ceiling of $20 does not make this possible.

The same choice would be made if the outlays for the two periods were combined.[3] In this case, the ranking would be made according to the ratios shown in row 5 in the table, and again the second project would be adopted. Contrast this to the situation in which there is an $18 budget ceiling for the first period and ample funds are provided to make both projects possible in the second period. In that event the ranking would be reversed by the recommended procedure: the ratios of row 6 prescribe adoption of the first project. However, the rule of choosing the set of projects that maximize total present worth clearly requires choosing the first project since the $30 present value of project 1 is larger than the $28 present value of project 2. Thus, it is possible to be misled by the single-period approach even where the budget constraint is actually binding for only one period.[4]

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[3]The outlays may be added since they are stated in terms of present values. This procedure is equivalent to averaging the outlays for the periods, as Lorie and Savage suggest.

[4]It is, of course, true, as the authors point out in the following section of their article (*loc. cit.*, p. 233), that the imposition of an additional constraint can never increase the value of the optimum. Hence, the indicated imposition of an $18 budget ceiling in Period 1, which causes a reversal in the rule of choice, suffices to establish this as a counterexample to the recommended Lorie-Savage procedure, albeit one based on the discontinuities they neglected. (See also an example of a similar situation presented by A. G. Hart in, "Anticipations, Uncertainty and Dynamic Planning," *Studies in Business Administration*, XI, No. 1 [Chicago: University of Chicago Press, 1940], p. 47.)

CHAPTER 3

A Linear Programming Approach to the Lorie-Savage Problem

The problem of allocating fixed budget dollars among competing investment proposals, which we have called the Lorie-Savage problem, has a structure highly suggestive of linear programming. The choice to be made is from among a large number of combinations of alternatives, each of which may be evaluated by a linear function. The main divergence from the strict linear programming format is that decisions about individual projects must be made on an all-or-nothing basis. Presumably, half a bridge of a given design is an inadmissible alternative. Similarly, several bridges are not necessarily desirable in proportion to their number, if, indeed, the additional ones are useful at all.

In this chapter we shall show that the Lorie-Savage problem may be solved by means of linear programming and that this formulation yields useful additional information. We shall argue that, although integer programming methods (to be analyzed in Chapters 4 and 5) may be used to deal rigorously with the indivisibility of investment projects, this aspect creates only a minor problem in the linear programming solution, which has the advantage of simplicity in computability and interpretation.

3.1. The Basic Model

We shall use symbols c_{ij} and C_i, as before, to denote the costs of

projects and the budget ceilings in year t, respectively, but we now let the present value of all revenues and costs associated with individual projects be denoted by b_j (instead of the y_j used by Lorie and Savage[1]). Further, we introduce the symbol x_j to represent the fraction of project j undertaken. Then a simple model for selecting among independent alternatives those projects whose total present value is maximum, but whose total outlay in each period falls within the budget limitation is:

$$\text{Maximize} \quad \sum_{j=1}^{n} b_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} c_{tj} x_j \leq C_t, \tag{3.1}$$

$$0 \leq x_j \leq 1.$$

This model accomplishes the following. The problem of indivisibilities is solved in the sense that the linear programming solution implicitly looks at all *combinations* of projects, not just one project at a time, to select that set whose total present value is maximum. Furthermore, the upper limit of unity on each x_j guarantees that no more than one of any project (such as, say, a rolling mill) will be included in the final program. The omission of such a limitation would clearly lead to allocating the entire budget to multiples of the "best" projects. What this model does not accomplish is the elimination of fractional projects from the solution since that would involve nonlinear restrictions requiring the x_j to be either zero or one.

3.2. Linear Programming Solution to the Lorie-Savage Example

The Lorie and Savage two-period example of the preceding chapter (Table 2.1[2]) is presented in Table 3.1 to illustrate this model. Inequalities (3.1) are converted to equations by the addition of "slack variables" which are also constrained to be nonnegative. The slack variables denoted by s_1 and s_2, which are present values of the unspent amounts in periods one and two, respectively, are shown in the first pair of constraints given in Table 3.1. Other slack variables, symbolized as $q_1, \ldots, q_9$, are also introduced to convert the constraints $x_j \leq 1$ to cor-

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[1] *Op. cit.*, p. 234.

[2] See p. 9.

TABLE 3.1. LINEAR PROGRAMMING MODEL AND SOLUTION FOR THE LORIE AND SAVAGE EXAMPLE

Maximize:

$$14x_1 + 17x_2 + 17x_3 + 15x_4 + 40x_5 + 12x_6 + 14x_7 + 10x_8 + 12x_9$$

Subject to:

$$12x_1 + 54x_2 + 6x_3 + 6x_4 + 30x_5 + 6x_6 + 48x_7 + 36x_8 + 18x_9 + s_1 = 50$$

$$3x_1 + 7x_2 + 6x_3 + 2x_4 + 35x_5 + 6x_6 + 4x_7 + 3x_8 + 3x_9 + s_2 = 20$$

$$x_1 + q_1 = 1 \qquad x_4 + q_4 = 1 \qquad x_7 + q_7 = 1$$

$$x_2 + q_2 = 1 \qquad x_5 + q_5 = 1 \qquad x_8 + q_8 = 1$$

$$x_3 + q_3 = 1 \qquad x_6 + q_6 = 1 \qquad x_9 + q_9 = 1$$

Solution:

$x_1^* = 1.0$	$q_1^* = 0$	$\mu_1^* = 6.77$	$\gamma_1^* = 0$
$x_2^* = 0$	$q_2^* = 1.0$	$\mu_2^* = 0$	$\gamma_2^* = 3.41$
$x_3^* = 1.0$	$q_3^* = 0$	$\mu_3^* = 5.0$	$\gamma_3^* = 0$
$x_4^* = 1.0$	$q_4^* = 0$	$\mu_4^* = 10.45$	$\gamma_4^* = 0$
$x_5^* = 0$	$q_5^* = 1$	$\mu_5^* = 0$	$\gamma_5^* = 29.32$
$x_6^* = 0.970$	$q_6^* = 0.030$	$\mu_6^* = 0$	$\gamma_6^* = 0$
$x_7^* = 0.045$	$q_7^* = 0.955$	$\mu_7^* = 0$	$\gamma_7^* = 0$
$x_8^* = 0$	$q_8^* = 1.0$	$\mu_8^* = 0$	$\gamma_8^* = 0.5$
$x_9^* = 1.0$	$q_9^* = 0$	$\mu_9^* = 3.95$	$\gamma_9^* = 0$
$s_1^* = 0$		$s_2^* = 0$	
$\rho_1^* = 0.136$		$\rho_2^* = 1.864$	

Total Present Value: \$70.27

responding equations.[3] These q_j, $j = 1, \ldots, 9$, are the differences between the proportions of projects accepted by the program and the upper limit on the x_j of one. That is, any $0 < q_j < 1$ means the corresponding x_j is fractional.

[3]The upper bounds on the x_j do not require explicit statement as restrictions in many computer codes currently in use. Bounding techniques such as those developed by Charnes and Lemke permit implicit statement of the bounds, thus saving computer memory for other restrictions. See A. Charnes and C. E. Lemke, "The Bounded Variables Problem," *ONR Research Memorandum*, No. 10 (Pittsburgh: Graduate School of Industrial Administration, Carnegie Institute of Technology, 1954); and also, G. B. Dantzig, "Upper Bounds, Secondary Constraints and Block Triangularity in Linear Programming," *Econometrica*, 23, No. 2 (April, 1955), pp. 174-83.

The solution to this problem is displayed in the lower portion of Table 3.1. (The μ's, ρ's and γ's of the table will be defined below. We shall henceforth also adopt the convention of starring the symbols for variables when they are at their optimal values.)

The linear programming solution calls for the adoption, *in toto*, of projects 1, 3, 4, and 9, and partial acceptance of projects 6 and 7 — 6 to the extent of 97 per cent (= 100 per cent minus $q_6^* = 3$ per cent), and 7 to the extent of 4.5 per cent (= 100 per cent minus $q_7^* = 95.5$ per cent). Since both s_1 and s_2 are zero in the solution, no funds of the budgets go unused. In the Lorie and Savage solution, in which projects are either accepted or rejected, the first period budget is utilized only to the extent of \$48 out of the total of \$50.[4] But, of course, this difference is accounted for by the appearance of fractional projects: \$48 $\times$ .0454 = \$2.18 is added to the Lorie-Savage selection (from project 7) and \$6 $\times$ .03 = \$0.18 is subtracted from it (via project 6).

3.3. Fractional Projects

In the strict linear programming formulation each budget will usually be fully used because of the possibility of undertaking fractional projects. Notice that, although six projects are in the optimal program, only two enter in fractional amounts. This is not a coincidence resulting from the particular numbers of the example. Rather, it is a fundamental property of the model and the method of solution[5] which can never introduce more fractional projects than the number of time periods for which budgets are stated. The proof of this important proposition is deferred to the final section of this chapter.[6] Instead, we introduce the following geometric proof as a basis for an intuitive grasp of what is proved rigorously later on.

Consider, for example, a one-period problem. In choosing from among all feasible combinations of entire projects (where feasibility means that the total expenditure on the set of projects is no greater than the limit imposed by the budget ceiling) we suppose that that set is selected which has the highest total present value attainable. Any remaining funds not employed in carrying out the chosen set of integral

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[4]Since all parameters of the model are present values, the value of the difference between \$50 available and \$48 used is a present value, realizable either by lending or not borrowing. See also the discussions of Chapters 8 and 9, below.

[5]For example, the Simplex method, which utilizes only basis sets of vectors.

[6]Section 3.8, below.

projects may be utilized on at most a fraction of one or more of the unselected projects. We may assume, without loss of generality, that all projects have net positive present values since, in any event, the linear programming procedures will discard those whose present value benefits are exceeded by the present value of their associated outlays.[7] Thus the budget will always be exhausted.[8] By construction, the set of complete projects chosen is the best; that is, it maximizes the total present value obtainable from the expenditure of the budgeted amount (or less) on entire projects. It will not be possible to add an entire project to utilize "left-over" funds but only one or more fractional projects.[9] In general, it will not be desirable to spend the remainder on more than one fractional project. The unselected proposals may be ranked by their ratios of present value to cost, and the unspent funds may be utilized entirely on a fraction of the best one. We may conclude that for a one-period problem there need be only one fractional project.

In a two-period problem it is possible that funds are left over in each of the two periods after the optimal set of whole projects has been selected. In such instances it may be that a fraction of a single additional project will optimally utilize all remaining funds in both periods. This will not, in general, be the case. Instead it may require fractions of two projects to use up the entire amount remaining in each period's budget, and the value of these fractions may be greater than the zero present value of "idle" funds. Once again, however, no more than two such projects are necessary to take up the slack completely.

We may obtain a graphical representation of this two-period case by adjusting the budgetary ceilings to distinguish between integral and nonintegral accepted projects as follows. Let C_1 and C_2 represent the originally budgeted amounts in periods 1 and 2, respectively. Then let the amounts of expenditure of the corresponding integral projects which have been accepted, be totalled and deducted from their respective budgets. The residual amounts, if any, are then reserved for allocation to, at most, fractions of projects.

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[7]Alternatively, these may be discarded at the outset unless there are interdependencies reflected in the constraints — for example, one project with large positive net benefits can be adopted only if another one with negative net benefits is adopted. Such cases will be dealt with in Section 3.6, below.

[8]Unless, of course, all projects with positive present values have already been accepted.

[9]An exception can arise if there are ties between any of the selected projects and the remaining ones. In that event a substitution can take place between entire projects or sets of projects. This does not alter the argument to follow.

Thus, if we designate by C_1' and C_2' as, respectively, the residual budgets in years 1 and 2 after performing the subtraction of all integral projects, we can plot them as horizontal and vertical lines by reference to the Cartesian coordinate system whose *vertical* axis is stated in terms of year 1 dollars and whose *horizontal* axis is stated in terms of year 2 dollars.

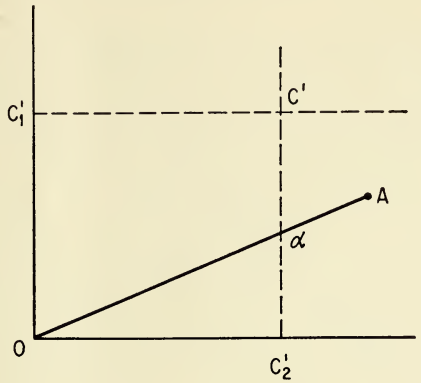

Figure 3.1.

This is done in Fig. 3.1. (See the broken lines.) Then we can use this diagram to analyze the status of any fractional projects that have also been adopted because these must come from either, or both, of the residual budgetary amounts C_1' or C_2'.

In Fig. 3.1 point A lies below the line C_1' and to the right of C_2'. Hence it represents a project whose complete (that is, integral) adoption would require more funds in the second period than still remain unallocated. Since it lies below C_1' the outlay required for this project in the first period is smaller than the appropriately altered budget ceiling, C_1'. Any point on the line $O\alpha$ which lies between O and A, however, represents a fraction of the total project A that can be adopted without violating either C_1' *or* C_2'. Among the admissible fractional parts of A, the largest one corresponds to α. If this fraction, α, of project A is adopted, however, then, as the diagram shows, only first period funds are left over. This is represented by $C' - \alpha$. But then, observe that both C_1' and C_2' were defined by reference to residual funds after the integral projects had been exhausted. Hence $C' - \alpha$ allows, at best, adoption of a fraction of an additional project.

In the less trivial case in which only projects remain whose outlays in both time periods require more than is still available, each such project would be represented by a point lying above and to the right of the point

C' in the diagram. To achieve this point would require the adoption of two projects, each fractionally (except coincidently if the point representing the best remaining project were to lie on a line through OC'). For clarity we have depicted this situation in the separate Fig. 3.2. Two

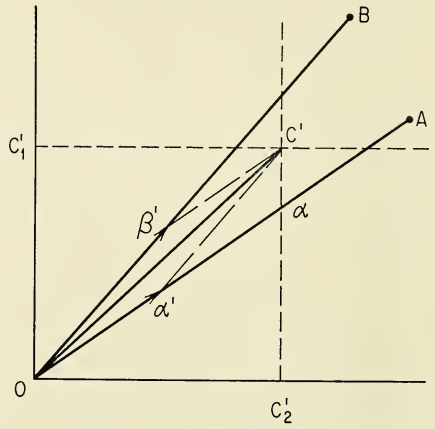

Figure 3.2.

projects, both of which require more than the funds remaining in the two periods, are represented by points A and B lying both above and to the right of the point C' in this diagram. In this instance the presence of two such projects makes it possible to utilize fully the budgets for *both* periods. A pair of nonnegative fractions α', β' satisfying $\alpha' A + \beta' B = C'$ will produce this result, as shown in the diagram.

A question naturally arises whether it is desirable to use up the remainder of the budget in each time period or to accept a fraction of only a single project. The answer depends on the present value of the increment of a second fractional project that is to be added as compared to the present value of that increment of the first fractional project which must be reduced to maintain feasibility. Suppose, for example, that the fraction α of project A were utilized in Fig. 3.2. This would exhaust the second period budget without leading to a violation of the first period budget constraint. But then, as is evident from the rules for parallelogram composition, none of project B could be undertaken and these funds, $C' - \alpha$, would remain idle. This may, of course, still represent the best use of the funds. If a positive fraction of project B is to be used to fill out the program, the use of project A must be reduced to α' while β' of

project B can be achieved. This substitution will pay only if

$$\alpha b_A < \beta' b_B + \alpha' b_A$$

or

$$\frac{\alpha - \alpha'}{\beta'} < \frac{b_B}{b_A}$$

where b_A and b_B represent returns, per unit adoption, on projects A and B, respectively. Except for ties there will be a preference among the remaining projects based on their present values and their outlays.

In Fig. 3.1 we have exhibited a case of a single project, A, in which some idle funds necessarily remain from the period 1 budget. If two such projects were to be considered that violated only the period 2 budget then, evidently (except for ties),[10] we would choose to allocate all remaining funds to the project yielding the best return. Similar remarks apply if a pair of projects violating only the C_1' constraint were presented for consideration. Only if at least one project such as A or B in Fig. 3.2 is present, however, is it possible to achieve a solution in which *all* funds for both periods are used. But even then, as we have just seen, it may not be desirable to utilize all funds, and only a fractional part of one of the projects may then be adopted.

All fractional possibilities for the two-period budget case have now been covered, and in none of them is it necessary to add more than two fractional projects to the collection of integral projects in order to obtain an optimal solution. This kind of diagrammatic representation is useful and has been supplied chiefly for purposes of insight. We have yet to synthesize a wholly rigorous and general proof that an optimum, if it exists, can always be achieved with at most T fractional x_j^* in a problem of the form (3.1) with $T + n$ constraints. Such a proof will be supplied,[11] but it is best deferred until we have developed certain further ideas which will enable us to treat problems involving mutually exclusive and contingent projects at the same time. Also, this will enable us now to turn to certain other aspects of linear programming, such as the relations of duality that are known to hold, and to exploit these in a way that will bring their managerial significance to the fore at the same time that we continue our analysis of the Lorie-Savage proposals.

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[10]Such ties still permit us to choose (arbitrarily) between the candidates and still achieve an optimum solution so that their presence does not affect our conclusions in any way.

[11]See *infra*, Sec. 3.8.

3.4. The Dual Problem and Its Interpretation

To every linear programming problem there corresponds another linear programming problem called its dual and generally much insight may be gained from an analysis of the dual problem. This is especially true here. The dual problem is

$$\begin{aligned} &\text{Minimize} && \sum_{t=1}^{T} \rho_t C_t + \sum_{j=1}^{n} \mu_j \\ &\text{Subject to} && \sum_{t=1}^{T} \rho_t c_{tj} + \mu_j \geq b_j, \qquad j = 1, \ldots, n \\ & && \rho_t,\ \mu_j \geq 0. \end{aligned} \tag{3.2}$$

Both the primal (direct) and the dual problem are placed into a single table, Table 3.2, in which the variables of the primal are written across the top and the variables of the dual along the left margin. The problem is stated in terms of the original set of inequalities as indicated by the symbols along the bottom and the right margin.

When displayed in this way one may recognize the meaning of the ρ_t as evaluators of the budget constraints. ρ_t^*, the value of ρ_t in the optimal solution, represents the present value of an additional dollar added to the budget in period t, assuming optimal use of the budgets. It is a marginal value in the sense that (a) it is an increment, and (b) it depends on optimal use of the indicated increment. If budgets are critical (that is, binding), it follows that the ρ_t^*, the "shadow" or opportunity costs of the budget constraints, must be nonzero.[12] However, the ρ_t^* will differ from period to period depending on the ability of the firm to utilize an additional dollar on the (currently) submarginal investments in each period.

The variables ρ_t refer to the budgetary limitations, C_t, in periods $t = 1, \ldots, T$. Another collection of dual variables is distinguished by the symbol μ_j. These are associated with the $j = 1, \ldots, n$ constraints $x_j \leq 1$ in the primal problem and also play a role as evaluators. By rearranging the restrictions of the dual problem, we may exhibit the μ_j^* more directly:

$$\mu_j^* \geq b_j - \sum_{t=1}^{T} \rho_t^* c_{tj}. \tag{3.3}$$

Since the μ_j are derived from the upper bound constraints on the projects, when a project is chosen so that, say, $x_j^* > 0$ obtains, the corresponding

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[12] If the budget for some period is not fully utilized, $\rho_t^* = 0$ is a condition of optimality.

TABLE 3.2. TABULAR REPRESENTATION OF THE PRIMAL AND DUAL LINEAR PROGRAMMING FORMULATION OF THE LORIE-SAVAGE PROBLEM

	x_1	x_2	x_3	$\cdots$	x_j	$\cdots$	x_n	
ρ_1	c_{11}	c_{12}	c_{13}	$\cdots$	c_{1j}	$\cdots$	c_{1n}	$\leq C_1$
ρ_2	c_{21}	c_{22}	c_{23}	$\cdots$	c_{2j}	$\cdots$	c_{2n}	$\leq C_2$
ρ_3	c_{31}	c_{32}	c_{33}	$\cdots$	c_{3j}	$\cdots$	c_{3n}	$\leq C_3$
.	.	.	.		.		.	.
.	.	.	.		.		.	.
.	.	.	.		.		.	.
ρ_t	c_{t1}	c_{t2}	c_{t3}	$\cdots$	c_{tj}	$\cdots$	c_{tn}	$\leq C_t$
.	.	.	.		.		.	.
.	.	.	.		.		.	.
.	.	.	.		.		.	.
ρ_T	c_{T1}	c_{T2}	c_{T3}	$\cdots$	c_{Tj}	$\cdots$	c_{Tn}	$\leq C_T$
μ_1	1	.	.		.		.	≤ 1
μ_2	.	1	.		.		.	≤ 1
μ_3	.	.	1		.		.	≤ 1
.	.	.	.		.		.	.
.	.	.	.		.		.	.
.	.	.	.		.		.	.
μ_j	.	.	.		1		.	≤ 1
.	.	.	.		.		.	.
.	.	.	.		.		.	.
.	.	.	.		.		.	.
μ_n	.	.	.		.		1	≤ 1
	$\geq$ b_1	$\geq$ b_2	$\geq$ b_3	$\cdots$	$\geq$ b_j	$\cdots$	$\geq$ b_n	

dual restriction is met exactly.[13] This means that the inequality above becomes an equation, which may be paraphrased, "The value of an accepted project is the excess, if any, of its present value, as calculated at the predetermined rate of discount, over the sum of the discounted outlays as evaluated at the opportunity cost of the limited funds of each budget year."

Utilizing a notation that is now standard in the linear programming literature[14] we may refer to $z_j - c_j$ values as (optimality) criterion values supplied by Simplex calculations in a linear programming tableau. For convenience let these values now be defined by $\gamma_j \equiv z_j - c_j$, as in

[13]See, for example, Charnes, Cooper and Henderson, *Introduction to Linear Programming* (New York: John Wiley and Sons, Inc., 1953), p. 28. See also the notion of "complementary slackness" in A. W. Tucker, "Dual Systems of Homogeneous Linear Relations," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems* (Princeton: Princeton University Press, 1956) pp. 15 ff.

[14]See, for example, Charnes, Cooper, Henderson, *loc. cit.*

Table 3.1, with $\gamma_j^* \geq 0$ necessarily obtaining at an optimum. Finally observe[15] that

$$z_j^* = \sum_{t=1}^{T} \rho_t^* c_{tj} + \mu_j^*$$

and so we may write

$$\gamma_j^* = \sum_{t=1}^{T} \rho_t^* c_{tj} + \mu_j^* - b_j. \tag{3.4}$$

With these definitions available we can now turn to a discussion of the rejected projects. With each x_j we have an associated slack variable, q_j, as in Table 3.1, given by the relation

$$q_j = 1 - x_j \geq 0. \tag{3.5}$$

If $q_j^* > 0$, then also $x_j^* < 1$, and this implies $\mu_j^* = 0$ so that (3.4) becomes

$$\gamma_j^* = \sum_{t=1}^{T} \rho_t^* c_{tj} - b_j \geq 0 \tag{3.6}$$

for these cases. If also $0 < x_j^* < 1$, that is, j is a (proper) fractional project, then (3.6) becomes

$$\gamma_j^* = \sum_{t=1}^{T} \rho_t^* c_{tj} - b_j = 0 \tag{3.7}$$

which means that,[16] in these cases, an accepted project (even if fractional) must fulfill the condition that its present value, b_j, equal the present value of all future expenditures,

$$\sum_{t=1}^{T} \rho_t^* c_{tj},$$

where the ρ_t^* represent the relevant interest rates (or cost of capital) obtained by an imputation that considers all available opportunities.

Projects like the fractional ones just examined, which necessarily imply the fulfillment of (3.7), may be regarded as truly marginal for acceptance. This condition can also, however, be fulfilled for the case $x_j^* = 0$, for example, when alternate optima are available — in which

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[15]See A. Charnes, and W. W. Cooper, *Management Models and Industrial Applications of Linear Programming* (New York: John Wiley and Sons, Inc., 1961), Chapter XV ff.

[16]Note therefore that any fractional project will have both μ^* and γ^* equal to zero.

event we can regard these as projects that are marginally rejected. In contrast, if

$$\gamma_j^* = \sum_{t=1}^{T} \rho_t^* c_{tj} - b_j > 0 \tag{3.8}$$

occurs then we must *necessarily* have the jth project totally rejected so that $x_j^* = 0$, necessarily, in such cases. These we may refer to as submarginal projects, judged by reference to alternate resource uses, since then the net present value of the benefits, b_j, is *less* than the present value of the expenditures (and foregone alternatives) that must be undertaken, as given by the optimizing values ρ_t^* in (3.8).

The μ_j^* and γ_j^* are by-products of the linear programming solution and are "natural" quantities for ranking all projects considered. The rank order so established may well differ from rankings by present value or by "internal rate of return." Its chief difference and virtue lies in that it takes into account not only the merits of the individual projects, but also their interrelationships via the budget constraints. The practical importance of ranking projects and the effect of budget constraints on ranks are subjects to which we shall return in the next two chapters. In the meantime we consider the relation of the dual problem to the Lorie-Savage method.

3.5. The Dual Variables and the Lorie-Savage Solution

When the Lorie-Savage method of solving this problem is compared to the dual problem, as represented by the system (3.2), a similarity may well be apparent. This similarity may be seen more clearly by placing the conditions on the optimal projects, that is, equations (3.3) and (3.6), alongside the criterion for acceptance of projects as laid down by Lorie and Savage. Their method, described in the preceding chapter,[17] consists of finding nonnegative quantities p_t such that

$$b_j - \sum_t p_t c_{tj} \tag{3.9}$$

is nonnegative for the accepted projects and nonpositive for the rejected ones.[18] The restrictions on the p_t given by expression (3.9) are not

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[17]See p. 9. We have restated their method in our notation, in which the b_j are the y_j of their notation. However, the p_t of their notation is retained.

[18]We have broadened their criterion to permit acceptance of projects for which expression (3.9) is zero, as must necessarily be admitted when alternate optima are present, and also allowed p_t to be zero, as when sufficient funds are budgeted in some years.

sufficient to make them unique.[19] In the preceding chapter we have exhibited a range of p values with the desired property for even a one-period problem.[20] Of course, for a linear programming attack, issues such as uniqueness for the ρ_t^* do not present any particular difficulty. Indeed, when the presence of alternate optimum solutions to a primal problem gives rise to nonuniqueness, additional and valuable by-product information may also be secured.

The relation between the p_t and the ρ_t^* may be clarified further by reference to a graphical procedure for the two-period Lorie and Savage example that was previously discussed. Such a graph may, in fact, have been intended by Lorie and Savage for determining the values of the p_t. In Fig. 3.3 each of the $j = 1, \ldots, 9$ projects (see Table 2.1) is represented by a line. Each line is associated with an equation

$$b_j - p_1 c_{1j} - p_2 c_{2j} = 0 \tag{3.10}$$

that is graphed by reference to horizontal and vertical axes that are labeled p_1 and p_2, respectively.

After the lines given by equations (3.10) corresponding to the projects of the Lorie-Savage example of Table 2.1 have been drawn on the graph of Fig. 3.3, the following graphical method may be employed to carry out the Lorie and Savage procedure for deciding on the set of acceptable investment proposals in the face of budget limitations in two years.[21] (The lines for individual projects may be identified by the numbers at their intercepts with the two axes; the projects accepted by the Lorie-Savage method are also marked with an asterisk.)

The procedure begins with the choice of an arbitrary point $P = (p_1, p_2)$ in the first quadrant of the diagram. Draw a ray OP from the origin through the point P such that this ray intersects the lines of all the projects as given by the expressions (3.10) after the assumed p_1, p_2 values have been inserted. Next, rank the projects in the order in which their lines are intersected by the ray OP, the highest rank (that is, no. 1) being given to the project whose line is intersected by ray OP *farthest* from the origin. After the projects have been ranked in this manner and listed, cumulate the outlays of the projects in each period. Tentative acceptance is indicated for all projects on the list up to that one whose

[19]There are n projects, and T values for p_t to be determined, which permits a multiplicity of solutions, if there are any, when $n > T$.

[20]See expression 2.5.

[21]Presentation of this procedure is not intended as a recommendation for its adoption. It is designed to illustrate properties of their method which are important to later discussions.

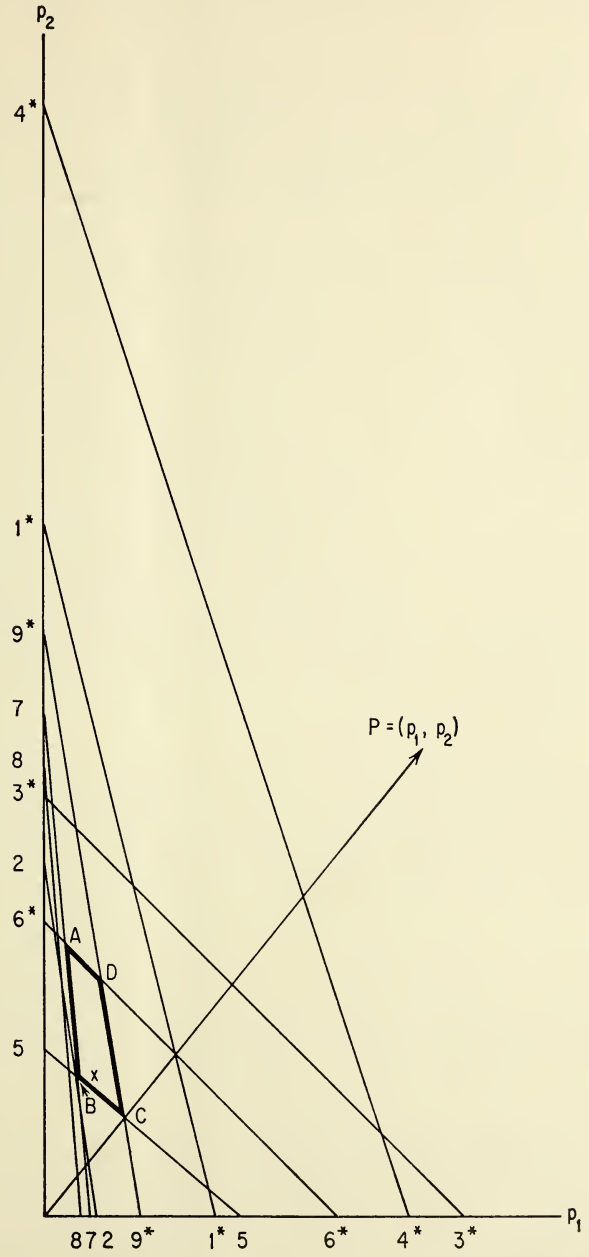

Figure 3.3.

inclusion would violate the budget constraint in at least one of the periods.

If the budget for the other time period is not fully exhausted by the list of tentatively accepted projects, improvement may be sought by the following iterative procedure. Move the trial point P away from the axis of the time period for which surplus funds are indicated and relatively closer to the axis corresponding to the time period for which the budget was previously the limiting factor. Repeat the procedure described above by drawing a ray from the origin through the new trial point P, rank the projects and cumulate their costs in each time period, thereby determining another tentative acceptance list. In this way seek a list of projects which comes "as close as possible" to exhausting both budgets.

As will be shown below, no simple rule can be laid down for ending this process. Apparently, however, this procedure should be continued as long as the list of accepted projects is being expanded; that is, projects are added while none are being dropped. The fact that a small movement of the point P does not alter the list of tentatively accepted projects is no guarantee that a larger movement will not do so. When alternative lists which do not fully utilize the budgets in both periods are compared, one may choose among them, as Lorie and Savage propose, by comparing the total present value of the lists of accepted projects. Alternatively, one may compare the values of

$$p_1 s_1 + p_2 s_2 \tag{3.11}$$

for the various lists, where the p_t are the coordinates of the trial points used to derive them, and the s_t are the amounts of the budgets not utilized by the listed projects in the given time period. By this alternate evaluation, one would choose the list for which quantity (3.11) is the smallest.

The values of p_1 and p_2 which result from this procedure may differ by a scale factor from those obtained by the method of Lorie and Savage. The final values of p_1 and p_2 they obtain are such that the dividing point between the accepted and rejected projects is $R = 0$ in the expression

$$R = b_j - \sum_t p_t c_{tj},$$

although other constant values for R might have served equally well. Performing the scaling of the values of p_1 and p_2 which result from the geometric procedure above to attain this property only moves the final point P along its ray OP.

When the geometric procedure, including scaling of the final values of p_1 and p_2, has been carried out for the present numerical example, a

quadrilateral containing the set of valid points P in its interior is obtained. The corner points of this region, which is drawn in bold lines in the diagram, are given by the intersections of lines representing projects 6 and 7 (point A); projects 5 and 7 (point B); projects 5 and 9 (point C); and projects 9 and 6 (point D). The values $p_1 = 0.33$ and $p_2 = 1.0$, chosen by Lorie and Savage, fall at the point indicated by the cross in the diagram. However, any other point inside the quadrilateral would have served as well.

When a ray is now drawn from the origin through the cross and extended outward beyond any of the lines shown in the diagram we then proceed, as before, by reference to successive intersections with the ray, starting from the one farthest away from the origin, continuing to the next farthest, and so on as long as the budget constraints admit the successive cumulations. This procedure designates the projects to be accepted as $j = 4^*, 3^*, 1^*, 6^*, 9^*$, in that order, and this designation is the same as the Lorie-Savage result which was given in Table 2.1.

As has just been mentioned, Lorie and Savage gave values $p_1 = 0.33$ and $p_2 = 1.0$ for their problem. How does this compare with the linear programming values? First, we should remark, of course, that the linear programming solution yields the values of the dual variables as a by-product so that there is no need to solve the dual problem explicitly. In the Lorie and Savage example the values for ρ_1^* and ρ_2^* are 0.136 and 1.86, respectively, and are located at the corner point A in the diagram. Thus the present worth of an additional dollar invested in the first period would be 13.6¢ while the (present) value of another dollar for the second period budget would be an additional $1.86. Note that these values apply to the linear programming solution in which there are two fractional projects. In the Lorie and Savage solution in which only the second period budget is exhausted, the value of an additional dollar for the first period would be zero.[22]

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[22]The discussion above serves to answer the Lorie and Savage comment (p. 234) that " . . . the multipliers p_1 and p_2 are auxiliary quantities for which there does not seem to be an immediate interpretation but that nonetheless help to solve the problem." The cryptic footnote which follows, "The multipliers, p_1 and p_2, are closely related to what are known in mathematics and in economics as 'Lagrange multipliers' " is also explained by the linear programming formulation by the well-known relations between dual variables and Lagrangian multipliers. See, for example, A. W. Tucker, "Linear and Non-linear Programming," *Operations Research*, V, No. 2 (April, 1957), pp. 244-57.

The apparent anomaly of "free" budget dollars will be discussed in some detail in connection with the dual variables of the integer programming model, in Section 5.3.

3.6. Mutually Exclusive and Contingent Projects[23]

So far the comparison between the Lorie and Savage solution and the straightforward linear programming approach has been made entirely for the case of independent projects. The programming approach is such, however, that a mixture of independent and mutually exclusive projects presents neither conceptual nor computational difficulties. To make the extension we merely add one equation for each set of mutually exclusive alternatives, namely,

$$\sum_{j \epsilon J} x_j \leq 1 \tag{3.12}$$

where the summation is over the indices corresponding to the set J of mutually exclusive alternatives. When different types of annealing furnaces are being evaluated, for example, the restriction states that the sum of all such furnaces will not exceed one. In this way at most one will be chosen. Note, also, that the addition of such a restriction makes a separate upper bound of unity on each x_j of the mutually exclusive set redundant. In computer codes in which upper bounds must be expressed as separate restrictions, this implies the elimination of at least as many inequalities, $x_j \leq 1$ as have been added by expressions like (3.12).[24] The critical problems of computation time and computer capacity are thus ameliorated.

Although restricting the optimal solution to accommodate the problem of mutually exclusive investments will never, for example, call for adoption of two complete furnaces, a question arises whether restriction (3.12) can lead to acceptance of several fractional furnaces. The answer is that it can, and when the time pattern of expenditures of the mutually exclusive opportunities are different (as, for example, when one furnace has a much lower initial cost, but requires much more frequent overhauling) fractional projects may appear in the optimal solution.

The possible existence of fractions of several mutually exclusive alternatives in the optimal linear programming solution is less disconcerting than it may seem to be at first glance. In the case of two basically different bridge designs, for example, the solution may appear to call for, say, two-thirds of a steel bridge and one-third of a wooden one. While construction of such an object is a patent absurdity, this outcome may be interpreted in a way that provides still further information on a

[23] See p. 11 for definitions.

[24] $\sum_{j \epsilon J} x_j \leq 1$ implies $x_j \leq 1$ for every $j \epsilon J$, and so the latter constraints on the individual variables become redundant and may therefore be dropped.

possible "optimal" bridge which may be better than either the steel or the wooden one. Thus, it may be a signal to the engineers to try a new design, less expensive than the original steel bridge, but more durable than the original wooden one. Where the number of possible alternative designs is large, this ability to use the fractional solution based on extremes as an indicator of a better compromise may well be an advantage rather than a drawback of the approach.

Furthermore, the number of fractional projects that can appear in the linear programming solution will still be small relative to the number of investment projects that may be included in any given problem. The proposition referred to on page 19 may be extended to the case of mutually exclusive and contingent projects. The number of fractional projects is still limited by the number of time periods in the model.[25]

The situation represented by contingent alternatives as defined earlier,[26] may be handled by an extension of the method for mutually exclusive projects. One need only construct compound projects which include the contingent alternatives. For example, if the projects under consideration are refrigerated warehouses for wholesale meat distribution and sausage manufacturing equipment, the mutually exclusive alternatives are the warehouse with and without the auxiliary manufacturing facilities.[27] Generating compound projects in this way is common in practice when the alternatives are few, although the multitude of possible ancillary activities of major projects discourages this approach. Fortunately it is possible to treat such related projects marginally and to permit the computer to generate the relevant combinations.

An alternate way of stating a contingency restriction would be the following. If project m is desirable only if project k is adopted, but not otherwise, then we require that

$$x_m \leq x_k. \tag{3.13}$$

If, in the solution, $x_k^* = 0$, that is, project k is rejected, then, necessarily, $x_m^* = 0$ also. In these expressions we require, as usual, that $x_k \leq 1$, but it is not then necessary to write a similar constraint for x_m since $x_k \leq 1$ and $x_m \leq x_k$ implies $x_m \leq 1$. Of course, the opposite implication need not hold here — namely, $x_k^* = 1$ need not imply $x_m^* = 1$. An additional possibility, ruled out under the compound project formulation, is that

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[25]See the theorem of Section 3.8.

[26]See p. 11.

[27]It may be presumed that sausage making would not be considered in the absence of the warehouse.

x_m^* is fractional even when $x_k^* = 1$ or that they are different fractions which satisfy relation (3.13).

More elaborate sets of alternatives may be stated in a similar way. "Nested" sets of contingent projects may arise as, for example, when a meat warehouse can be equipped with sausage manufacturing equipment and also with hickory smoking facilities. The mutually exclusive alternatives will then be (a) the warehouse itself, (b) the warehouse with sausage equipment, and (c) the warehouse with both sausage equipment and smoking facilities. If the alternate method of expressing the restrictions is used, the warehouse, the sausage equipment and the smoking facilities would remain separate projects, but the restrictions would be

$$x_b \leq x_a, \qquad x_c \leq x_b, \qquad x_a \leq 1.$$

This situation is similar to one in which two mutually exclusive contingent projects compete, such as a building equipped with escalators versus one equipped with elevators. Here the restrictions to be added would be

$$x_b \leq x_a, \qquad x_c \leq x_a, \qquad x_b + x_c \leq 1, \qquad x_a \leq 1.$$

A numerical example to illustrate some of these devices is presented in the appendix to this chapter.

3.7. Forcing Project Acceptance

One final variation worth mentioning here concerns the forced inclusion of some particular investments. For example, a company may decide (whether wisely or not) to build a certain facility regardless of its relative economic merit. If the project is an "independent" one, its cost in each period can be deducted from the budget before selection of the remaining proposals. When the choice of the "forced" investment is from among mutually exclusive projects, such netting out of the anticipated costs is not possible. Within the linear programming framework one of the set of mandatory investments can be forced simply by the elimination of the slack variable of their upper bound constraint. The particular alternative chosen, if a solution exists, is then consonant with the different budgets and with the other sets of proposals available.

The versatility, or at least the promise, of the proposed mathematical programming approach to the Lorie and Savage problem (allocating fixed budgets for several time periods among competing investment alternatives) has by now been sufficiently demonstrated, at least in

principle. The investment projects need not be independent but may include sets of mutually exclusive projects as well as contingent sets of investments simultaneously. Even the prior selection by which the projects must have positive net present values as calculated at the appropriate discount rate becomes unnecessary, though, perhaps, desirable from a computational point of view. We shall see that the Lorie and Savage problem is an integer programming problem, but one which can be handled more advantageously, with regard to the problem of computation, as a linear programming problem despite the possibility of fractional projects in the solution. That this disadvantage is limited is shown in the theorem alluded to before, and which is proved in the next section.

3.8. Theorem on the Maximum Number of Fractional Projects in the Linear Programming Solution to the Lorie-Savage Problem[28]

A. All investment projects are independent.

Theorem. The maximum number of fractional projects in any basic solution is equal to the number of periods, T, for which budget constraints are imposed.

Proof.[29] We assume that the constraints

$$\sum_{j=1}^{n} c_{tj}x_j \leq C_t, \qquad t = 1, \ldots, T$$
$$x_j \leq 1, \qquad j = 1, \ldots, n$$
$$x_j \geq 0, \qquad j = 1, \ldots, n$$

admit only bounded solution sets. We can then replace any such constraint set with the following equivalent, whose structure will facilitate exposition:

$$\begin{aligned} &\text{(a)} \quad \sum_{j=1}^{n} c_{tj}x_j + s_t \qquad\qquad = C_t, \qquad t = 1, \ldots, T \\ &\text{(b)} \quad x_j \qquad + x'_j \qquad\quad = 1, \qquad j = 1, \ldots, n \\ &\text{(c)} \quad s_t \qquad + s'_t = U, \qquad t = 1, \ldots, T. \end{aligned} \tag{3.14}$$

[28]The particular method of proof was chosen for the additional insights it presents and for its extension to related situations. Simpler proofs for this part can be easily constructed.

[29]See A. Charnes and C. E. Lemke, *op. cit.*, for the original statement of this kind of proof in their development of the so-called bounded variables techniques in linear programming.

Here, as usual, all variables are constrained to nonnegative ranges. U, the stipulation assigned to the T constraints on the slack variables of the budgets, is an unspecified but suitably large upper bound so that, for any bounded solution, we will have $s'_t = U - s_t$ with $s_t \ll U$ so that only values of s'_t will involve U,[30] and, therefore, none of the original slack possibilities for any s_t will be disturbed because of these new constraints.[31]

The structure of (3.14) has been written in a way that brings certain features to the fore. Evidently there is a certain pairing of the primed and unprimed variables. The total number of variables is $2(T + n)$ while the number of constraints (apart from nonnegativity) is $2T + n$. Because of the positioning of the nonzero coefficients for these variables, we see, immediately, that any basis set formed from the matrix of these coefficients will have exactly $2T + n$ variables in it. Also, the variables associated with any such basis will have the following property. Either an x_j or an x'_j, or both, must be in the basis collection, and, similarly, either an s_t or an s'_t, or both, must also be members of the basis set.

Now consider any feasible basis formed from this collection and let the corresponding variables be partitioned as follows:

Group I: N variables, x_j and s_t, in the basis set whose corresponding x'_j and s'_t (also N in number) are also basic variables.

Group II: M variables, x_j and s_t, in the basis set whose corresponding x'_j and s'_t (also M in number) are *not* basic variables.

Group III: K variables, x_j and s_t, which are *not* in the basis collection and whose associated x'_j and s'_t (also K in number) are basic variables.

Note that only x_j values from Group I can satisfy the fractional requirement $0 < x_j < 1$ and that the number of x_j variables in this group cannot exceed N. Hence if $N = T$ the theorem is proved.

The number of nonbasic variables is

$$n = 2(T + n) - (2T + n)$$

and these must all come from Group II or Group III since, by virtue of the remarks made on the structure of (3.14), we know that at least one member of an x pair or an s pair must be in a basis collection, and only Group I utilizes *both* members of each pair. Hence, we have $M + K = n$.

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[30]That is, we are borrowing from the so-called regularization techniques of linear programming as developed and set forth in A. Charnes and W. W. Cooper, *Management Models, op. cit.*, Chapters VI and XII.

[31]Unless, of course, infinite optima (which are not at issue here) can appear in the original problem.

Also the requirements for a basis collection gives

$$2N + M + K = 2T + n$$

and since $n = M + K$, it follows that $N = T$. Q. E. D.

From this development we now also have the following corollary for interdependent (mutually exclusive or contingent) projects:

Corollary. When attention is restricted only to basis sets, then the maximum number of fractional projects is equal to the number of periods, T, for which budget constraints are fixed plus the number of constraints expressing relations of interdependence (mutual exclusion or contingency) between designated variables.

Proof. Any constraint that expresses the relation of mutual exclusion is of the form

$$\sum_{j \epsilon J} x_j + v = 1$$

where J refers to the set of indices for projects where mutual exclusion is specified, and where $v \geq 0$ is a slack variable associated with this constraint. Also the constraint for a contingency relation between any projects, r and s, may be written

$$x_r - x_s + v = 0.$$

The argument utilized in the proof of our theorem assumed only the solvability and boundedness (for optimization) of the original constraint set. There was, in particular, no other specialization of the values for any c_{tj} or C_t. Hence, to prove this corollary, we only need to interpret all relations of mutual exclusion or contingency as new "budget constraints" and to assume that they do not introduce any inconsistency into the system.

Of course, the number of budgetary constraints plus the added relations of contingency and mutual exclusion may be uncomfortably large in certain cases of real application. In contrast, the theorem and its corollary above refer only to a maximal number of fractional projects. Furthermore, as we observed earlier, any relation of mutual exclusion makes it possible to simplify the constraint set because

$$\sum_{j \epsilon J} x_j \leq 1$$

and

$$x_j \geq 0$$

together imply $0 \leq x_j \leq 1$ for all $j \epsilon J$ so that the latter collection becomes redundant and may be removed from further explicit considera-

tion. Alternatively, if some $x_j = 1$ and all other $x_k = 0$, $j, k \in J$, before the imposition of a constraint that implies mutual exclusion for the members of J then this constraint is redundant since it is automatically satisfied by an optimum for a less restricted problem. Similar remarks also apply to the constraints $x_r - x_s \leq 0$ except here, of course, we admit both $x_r = x_s = 1$ and $x_r = x_s = 0$, if achieved before the imposition of this constraint.

These remarks are intended to suggest a computing strategy that may often simplify matters. If the constraints for mutual exclusion and contingency are not too numerous, they may be omitted from the initial problem for which calculations are undertaken. After an optimum has been achieved for this less constrained problem that also satisfies the integer requirement it may then be checked to see if it also satisfies the omitted constraints. If this is the case, then an optimum for the full problem is at hand since the latter involves more constraints and therefore cannot have a better optimum than the problem that was solved. In contrast, if satisfaction of some of the (originally) omitted constraints is not achieved in this way, it is not necessary to commence *ab initio.* Instead, the unsatisfied constraints may be adjoined directly to the already achieved tableau, by any of the techniques that are available for this purpose,[32] and by recourse, for example, to Lemke's dual method,[33] the calculations may be iterated until a new optimum is achieved that does satisfy these constraints.

We conclude this section by observing that experience with computations over a variety of problems of the indicated kind suggests that the number of fractional projects which will occur in an optimal solution is usually considerably smaller than the maximum that this theorem and its corollary admits. Because the computer codes that deal with programs involving integer solution requirements are not now in a very satisfactory state, this is all to the good. However, these theorems appear to represent about all that can now be said with real precision about integer solution requirements in these problems and the possibility of a maximal number of fractional values must be allowed for, at least until sharper results and a better understanding of the integer programming procedures is available.[34]

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[32]See, for example, Charnes and Cooper, *Management Models*, *op. cit.*, Chapter V ff.

[33]This and other variants that are available in various computer codes are described in Charnes and Cooper, *ibid.*, Chapter XIII.

[34]Note, for instance, this theorem and its corollary refer to *any* feasible basis so that there may exist a possibility for further improvement, at least in certain classes of problems, when attention is restricted only to optimal bases.

APPENDIX 3A

Numerical Example of the Treatment of Contingent Projects

The following example is designed to illustrate the linear programming formulation and solution to a Lorie-Savage type problem involving mutually exclusive and contingent projects. Two alternative methods will be employed to handle the contingency relationship between projects. The data for the example are presented in Table 3A.1.

TABLE 3A.1. HYPOTHETICAL DATA FOR THE NUMERICAL EXAMPLE

Project	Net Present Value	Present Value of Outlays			Headings Where These Data Appear	
		Period 1	Period 2	Period 3	Table 3A.2	Table 3A.3
a	20	24	16	6	P_1	P_1
b	16	12	10	2	[a]	P_2
c	11	8	6	6	[a]	P_3
d	4	6	4	7	P_2	P_4
e	4	1	6	9	P_3	P_5
f	18	18	18	20	P_4	P_6
g	7	13	8	—	P_5	P_7
h	19	14	8	12	P_6	P_8
i	24	16	20	24	P_7	P_9
j	4	4	6	8	P_8	P_{10}
Budget Ceilings		$C_1 = 50$	$C_2 = 30$	$C_3 = 30$	P_0	P_0

[a]These are components of composite projects, as discussed below, that are shown under $\hat{P}_9$ and $\hat{P}_{10}$.

The projects listed in Table 3A.1 are to be regarded as independent with the following exceptions. While project g is independent of all others, acceptance of project b is contingent upon acceptance of project g. Similarly, while projects a and d are to be regarded as independent, project c is acceptable only upon acceptance of *both* projects a and d.

In the first treatment of these relationships, as given in the initial linear programming tableau of Table 3A.2, the contingency relationships are expressed by means of composite projects which combine the flows associated with the dependent and the independent project. Thus, composite project $\hat{P}_9$ represents acceptance of both projects b and g, and the present values of their outlays in years 1, 2 and 3 are respectively combined; that is, the outlays of \$25, \$18 and \$2 in years 1, 2 and 3 for project $\hat{P}_9$ result from adding together \$12 + \$13 = \$25, \$10 + \$8 = \$18, and \$2 + \$0 = \$2, the outlays for projects b and g in each of these years. Similarly, the net present value of composite project $\hat{P}_9$ is the sum of the net present values of projects b and g, \$16 + \$7 = \$23. The contingency relationship between projects g and b is then expressed by restriction S_1 in Table 3A.2, in which the sum of the acceptable fractions of project P_5 (project g) and composite project $\hat{P}_9$ (projects b and g) is restricted to a maximum of 1. In effect, projects P_5 and $\hat{P}_9$ have been made mutually exclusive by this restriction.

Similar treatment is accorded projects a, d and c in expressing the second contingency relation. A composite project $\hat{P}_{10}$ is constructed which combines the net present values and the present values of the outlays in each of the years for the three projects. Thus, the net present value of project $\hat{P}_{10}$ of \$35 is the sum of the net present value of project a (\$20), project d (\$4), and project c (\$11), and the present values of the outlays of project $\hat{P}_{10}$ of \$38, \$26 and \$19 are the sums of the outlays of the three projects, a, c and d in each of the three years, namely, \$38 = \$24 + \$8 + \$6, \$26 = \$16 + \$6 + \$4, and \$19 = \$6 + \$6 + \$7. The dependence of project c upon projects a and d is then expressed by making both project P_1 (a) and composite project $\hat{P}_{10}$, and project P_2 (d) and composite project $\hat{P}_{10}$ mutually exclusive by restrictions S_2 and S_3. By virtue of the first of these restrictions, project P_1 (a) and composite project $\hat{P}_{10}$ are made mutually exclusive, requiring either the adoption of independent project a alone, or of all three projects, a, d and c or none of them. The second of these two restrictions, which must hold simultaneously, makes project P_2 and composite project $\hat{P}_{10}$ mutually exclusive, requiring the adoption of either the independent project d, or of all three projects, a, d and c, or none of them.

In the alternate method of stating contingency relationships, as portrayed in the initial tableau of Table 3A.3, the dependent projects are treated as marginal additions to the independent ones. Although this results in slightly weaker conditions when fractional projects appear, the approach yields exactly the same results as the treatment above when integer solutions are obtained. Under the alternative approach the first of the two contingency relations is

TABLE 3A.2. INITIAL LINEAR PROGRAMMING TABLEAU AND SOLUTION FOR THE FIRST TREATMENT OF THE NUMERICAL EXAMPLE

b_j		20	4	4	18	7	19	24	4	23	35						
Vector	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	$\hat{P}_9$	$\hat{P}_{10}$	R_1	R_2	R_3	S_1	S_2	S_3
R_1	50	24	6	1	18	13	14	16	4	25	38	1					
R_2	30	16	4	6	18	8	8	20	6	18	26		1				
R_3	30	6	7	9	20		12	24	8	2	19			1			
S_1	1					1				1					1		
S_2	1	1									1					1	
S_3	1		1								1						1
[a]																	
Solution	\$48.62						1				0.85	3.85		1.92			

[a]Upper bounds on P_3, P_4, P_6, P_7 and P_8 are omitted.

TABLE 3A.3. INITIAL LINEAR PROGRAMMING TABLEAU AND SOLUTION FOR THE SECOND TREATMENT OF THE NUMERICAL EXAMPLE

b_j		20	16	11	4	4	18	7	19	24	4					
Vector	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	R_1	R_2	R_3	S_1	S_2
R_1	50	24	12	8	6	1	18	13	14	16	4	1				
R_2	30	16	10	6	4	6	18	8	8	20	6		1			
R_3	30	6	2	6	7	9	20		12	24	8			1		
S_1			1					−1							1	
S_2		−1		2	−1											1
[a]																
Solution	\$48.63	.79		.89	1.0				1.0			3.89		0.89		

[a]Upper bounds on P_1, P_4, P_5, P_6, P_7, P_8, P_9, P_{10} are omitted.

expressed by the restriction

$$x_2 - x_7 \leq 0,$$

or,

$$x_2 \leq x_7.$$

Using the headings from the last column of Table 3A.1, one may interpret this restriction as saying that the fraction of project b adopted must not be less than the fraction of project g adopted. If both of these fractions are either zero or one, this implies that project b may be adopted ($x_2^* = 1$) only if project g is adopted ($x_7^* = 1$) although project g may be adopted even though project b is rejected ($x_2^* = 0$). Restriction S_2 in Table 3A.3 may be interpreted in similar fashion. After transposing

$$-x_1 + 2x_3 - x_4 \leq 0$$

to

$$2x_3 \leq x_1 + x_4$$

we may interpret this restriction in the following way. Since both x_1 and x_4 are also constrained to be less than unity, the effect of the restriction, if all the x_j are integers, is to permit either $x_1^* = 1$, or $x_4^* = 1$, or both (implying adoption of either project a or project b or both), but $x_3^* = 1$, implying adoption of project c, may obtain only if both $x_1^* = 1$ and $x_4^* = 1$. Of course, $x_1^* = x_4^* = 0$, that is, rejection of the independent projects, requires $x_3^* = 0$, or rejection of the dependent one.

In ordinary linear programming the variables x_j cannot be constrained to be integers, and hence in these models it may occur that some of the projects will be fractionally accepted. This does, in fact, happen in the solution to these alternative formulations of the same problem. In Tables 3A.2 and 3A.3, the solutions are indicated in the last row and may be read off the tables by associating the value in the "Solution" row with the project P_j of the given column. Thus, $x_6^* = 1$ indicates total acceptance of project P_6 (project h from Table 3.3), and $x_{10}^* = .85$ indicates that composite project $\hat{P}_{10}$ is accepted to the extent of 85 per cent. Since this project consists of projects a, d and c together, the solution calls for acceptance of only 85 per cent of each of these.[1]

The solution to the alternative formulation, given in the last row of Table 3A.3, is slightly different because of the presence of fractional projects in the optimal solution. In this version, project h is again accepted *in toto* ($x_8^* = 1$), but project d is now totally adopted ($x_4^* = 1$), while project a is accepted only to the extent of 79 per cent ($x_1^* = .79$), and the dependent project c is accepted to the extent of 89 per cent ($x_3^* = .89$). These values satisfy constraint (3.14),

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[1]Where fractional projects cannot be dealt with by adjustment of budgets or by other means, integer programming methods proposed in the next chapters must be resorted to.

although they do not quite express the intent of the constraint. The difference also brings about a slight change in the program value, from \$48.62 to \$48.63. A slight reduction in the amount of funds employed occurs in period 1 (the residual, R_1, goes from \$3.85 in Table 3A.2 to \$3.89 in Table 3A.3) while an increase in the funds employed in period 3 takes place (the residual R_3 goes from \$1.92 in Table 3A.2 to \$0.89 in Table 3A.4). The budget of \$30 in the second period is exhausted in both formulations, this amount being the limiting factor on project acceptance. Because funds availabilities are greater than requirements, the value of the surplus budget dollars in periods 1 and 3 is zero.

These two alternative versions of expressing contingency relationships in the linear programming formulation of this problem indicate the kind of problems that may be encountered as well as some of their implications. It should be pointed out, however, that where fractional solutions occur only with respect to independent projects, or where integer solutions are obtained by means of the methods of the next chapter, these problems do not arise. Hence the dependence will be fulfilled exactly, and the method may be chosen which best suits the circumstances.

CHAPTER 4

Integer Programming Solution to the Lorie-Savage Problem and Some Implications of Discreteness

In the previous chapter we have examined the Lorie-Savage problem and found that it could be formulated as a linear programming problem. This formulation was seen to have certain limitations, however, with respect to the handling of indivisibilities of investment projects. Recently a significant advance known as integer programming has been achieved that, among other things, permits the formulation and solution of the Lorie-Savage problem in full generality without these limitations. In this chapter we shall present this more general formulation and examine its major implications for the problems at hand.

4.1. Integer Programming Formulation of the Lorie-Savage Problem

Many problems heretofore intractable by mathematical programming or other methods now show promise of solution[1] by means of integer

[1]See, for example, G. B. Dantzig, "On the Significance of Solving Linear Programming Problems with Some Integer Variables," *Econometrica*, XXVIII, No. 1 (January, 1960), pp. 30-44.

programming, a technique for solving linear programs with the additional requirement that some or all of the variables in the solution take on only integer values.[2]

The algorithms of Gomory utilize straightforward linear programming routines for computation — for example, the Simplex and the Dual methods — but supplement these by generating new restrictions in the form of "cutting planes" that are applied to the original set of restrictions. Briefly, the cutting plane approach involves the addition of linear restrictions, supplementary to those of the original linear programming problem, that cut away part of the original feasible region without disturbing any of the original feasible "lattice points."[3] By successive cuts it is thereby possible to produce, in a finite number of steps, a new linear programming problem whose optimal solution is in integers. Since none of the original feasible lattice points has been cut away, the optimal solution to this augmented problem is also the optimal integer solution to the original problem. Some numerical problems are solved by means of two of the Gomory algorithms in the next chapter.

Two essential points are worth stressing with respect to these methods. First, the original linear programming problem to which the integer conditions are applied is replaced by another linear programming problem whose optimal solution is the solution to the original problem in integers. Second, it is not possible, in general, to state the restrictions of the new problem at the outset. Rather, these restrictions are generated during the computational process, and many of these added constraints may turn out, in the end, to be redundant. The choice of additional constraints therefore depends upon the route taken for the computations as well as on the initial formulation. "In principle," the method of

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[2]See three papers by R. E. Gomory: "Outline of an Algorithm for Integer Solutions to Linear Programs," *Bulletin of the American Mathematical Society*, LXIV (September, 1958), pp. 275-8; "An Algorithm for Integer Solutions to Linear Programs," *IBM Mathematics Research Project Technical Report*, No. 1, (Princeton: Princeton University, November, 1958); "All-Integer Integer Programming Algorithm," *IBM Research Report*, RC-189 (Yorktown Heights: International Business Machines Corp., January, 1960). See also E. M. L. Beale, "A Method of Solving Linear Programming Problems When Some But Not All of the Variables Must Take Integral Values," *Statistical Techniques Research Group Technical Report*, No. 19 (Princeton: Princeton University, July, 1958); A. H. Land and A. G. Doig, "An Automatic Method of Solving Discrete Programming Problems," *Econometrica*, XXVIII, No. 3 (July, 1960), pp. 497-520; and H. M. Markowitz and A. S. Manne, "On the Solution of Discrete Programming Problems," *Econometrica*, XXV, No. 1 (January, 1957), pp. 84-110.

[3]Points whose coordinates are integers.

Gomory cuts solves the problem of integer programming, but much remains to be done in order to improve its computational properties and work of this kind is under way — for example, improved methods for designating the pivots, and like devices, are being studied.

As was indicated in the preceding chapter, integer programming may be applied directly to the Lorie-Savage model to produce the optimal set of integral investment projects under budget constraints for several time periods. The model is seen to be virtually the same as that of equations (3.1), namely,

$$\begin{aligned} &\text{Maximize} && \sum_{j=1}^{n} b_j x_j \\ &\text{Subject to (a)} && \sum_{j=1}^{n} c_{tj} x_j \leq C_t, \qquad t = 1, \ldots, T \qquad (4.1) \\ &\text{(b)} && 0 \leq x_j \leq 1 \\ &\text{(c)} && x_j \text{ integer.} \end{aligned}$$

In the form of an integer programming problem, this model accomplishes all the requirements sought in the Lorie-Savage formulation. Integer programming, like linear programming, implicitly looks at all *combinations* of projects, not just one project at a time. The requirement that the x_j be integers together with the bounds placed on the x_j makes each of these variables into an "either/or," "go/no-go" variable which implies that a project is either accepted (that is, $x_j^* = 1$) or it is rejected (that is, $x_j^* = 0$).[4] Integer programming is therefore particularly suitable when there are additional constraints expressing contingency relationships between projects and when it is necessary to choose between projects in sets of mutually exclusive investment alternatives.

An integer programming solution to the example of Lorie and Savage was, in fact, presented in Table 2.1 above[5] and this solution is identical with that arrived at by Lorie and Savage. It calls for the acceptance of projects 1, 3, 4, 6 and 9, and yields a total present value of $70. All of the budgeted funds of the second period are required for these projects, although $2 of the first period budget remain uncommitted.

The success of the Lorie-Savage method in application to their own example does not, unfortunately, establish the validity of their procedure for all problems of this type. Indeed, it is in analyzing the failure

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[4]It is thus impossible for one-third of a bridge to appear in the optimal solution, provided, of course, that the stated problem has a solution in integers.

[5]See p. 9.

of their procedure in the face of indivisibilities that we are able to gain insight into the integer programming problem, and especially into the interpretation of the dual variables to the linear programming problem when integer conditions are imposed on the primal problem.

4.2. Failure of the Lorie-Savage Method

Since the Lorie-Savage procedure was described in detail in Section 2.2 and a geometric method equivalent to their method was developed in Section 3.5 there is no need to repeat the steps here. Instead we will simply show that if the Lorie-Savage example is augmented by another project then there may exist *no* parameters such that the quantity

$$y_j - p_1 c_{1j} - p_2 c_{2j} \tag{4.2}$$

is positive (or nonnegative) for the accepted projects of the optimal set, and negative for the remainder.

Consider a tenth project for the problem cited, with present worth of \$15 and discounted outlays of \$6 and \$7 in periods 1 and 2, respectively. The integer programming solution to the original problem calls for acceptance of projects 1, 3, 4, 6 and 9. The same projects are also designated for acceptance even after the new opportunity is made available.

Such is not the case, however, if we follow the procedure suggested by Lorie and Savage. This may be seen by reference to Fig. 4.1, which is the same as Fig. 3.3 except that a dashed line,

$$15 - 6p_1 - 7p_2 = 0$$

associated with the additional tenth project, is introduced. Observe that this dashed line dominates the solid line for project 6, among others, everywhere in the positive quadrant. That is, every ray from the origin to a point in the positive quadrant will intersect this dashed line farther from the origin than the line for project 6. Accordingly, by the procedure elaborated in Section 3.5, project 10 would be ranked higher than project 6 in every possible ranking; yet substitution of project 10 for project 6 would cause the second period budget constraint to be violated.

In Fig. 4.1 the quadrilateral $ABCD$ bounds the set of points whose coordinates are the values of p_1 and p_2 which have the desired property of separating the accepted from the rejected projects. That is, these values of p_1 and p_2 are such that expression (4.2) is nonnegative for the accepted projects of the original example and negative for the rejected ones. It will be seen that the same region will not enclose points which

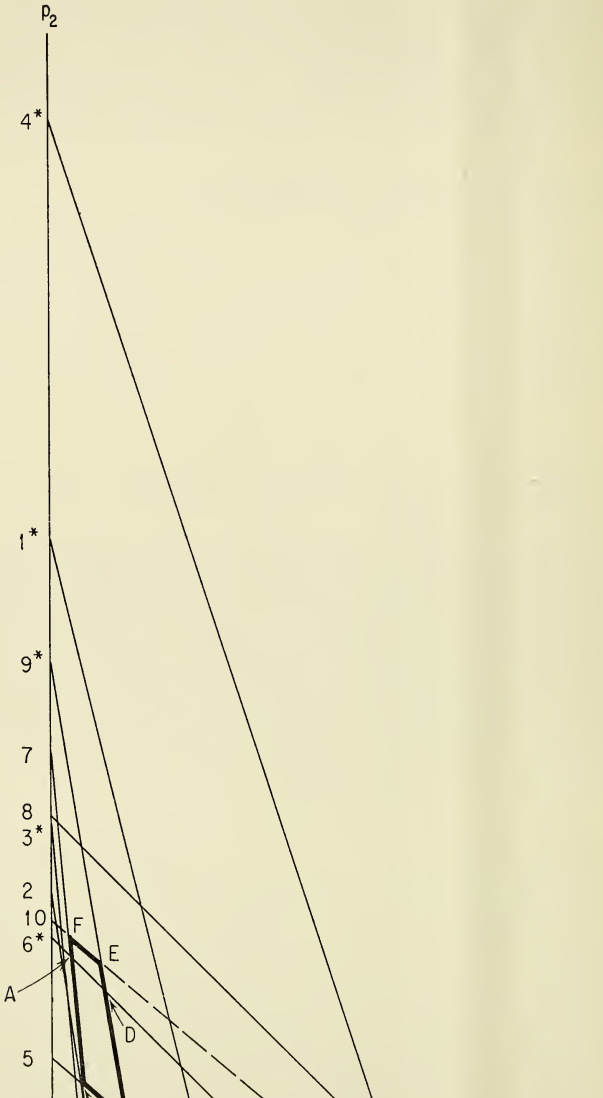

Figure 4.1.

have the desired property for the altered problem, since all points within it lead to the acceptance of both projects 6 and 10, in addition to the other projects of the earlier solution. The quadrilateral $ADEF$ does not accomplish our purpose either since it encompasses points which call for the acceptance of project 10 in place of project 6, thereby violating the budget constraint for the second period. In fact, because of the dominance of project 6 by project 10, no region can be found which will contain a point with the desired property; that is, no nonnegative values of p_1 and p_2 exist for which expression (4.2) is nonnegative for the accepted projects, and nonpositive for the rejected ones.[6]

The parallel between the dual variables of the linear programming solution and the Lorie-Savage method was shown earlier.[7] From that discussion it is clear that the existence of suitable values of p_t is connected with the existence of dual variables of the integer programming problem. While there is no difficulty in locating program values in the primal problem when integer conditions are imposed, a problem can arise with respect to the dual evaluators. This subject is extremely complex and has some important consequences. Hence we shall devote the entire next chapter to it. However, to facilitate this discussion we backtrack momentarily at this point in order to review certain aspects of the linear programming formulation.

4.3. The Linear Programming Model Without Upper Bounds

Consider the following linear programming problem:

$$\begin{aligned} &\text{Maximize} && \sum_{j=1}^{n} b_j x_j \\ &\text{Subject to} && \sum_{j=1}^{n} c_{tj} x_j \leq C_t, \qquad t = 1, \ldots, T \qquad (4.3) \\ & && x_j \geq 0, \qquad j = 1, \ldots, n. \end{aligned}$$

This problem resembles the earlier linear programming model but differs in that investments x_j are not defined in terms of "projects." By omitting the upper bounds of unity on the x_j we assume instead that there are constant costs and constant returns to scale to each project and that the b_j and c_{tj} are the unit discounted returns and costs. The realism of these assumptions need not concern us for the moment.

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[6]A nongeometric verification of this remark is presented in Section 5.8 below.
[7]See Section 3.5.

Any basic solution to the above problem will involve at most T nonzero values for the x_j. We can see this more directly in terms of the dual problem to the above:

$$\begin{aligned} &\text{Minimize} && \sum_{t=1}^{T} \rho_t C_t \\ &\text{Subject to (a)} && \sum_{t=1}^{T} \rho_t c_{tj} \geq b_j, \qquad j = 1, \ldots, n \qquad (4.4) \\ &\text{(b)} && \rho_t \geq 0, \qquad t = 1, \ldots, T. \end{aligned}$$

The optimality conditions give, for any particular j,

$$\sum_{t=1}^{T} \rho_t^* c_{tj} = b_j \qquad (4.5)$$

whenever $x_j^* > 0$. Hence these values of ρ_t^* fulfill constraints (4.4a) exactly — that is, as *equations* — for activities whose optimal x_j^* are greater than zero in the primal problem, while weak inequality holds for activities whose x_j^* are zero.[8] This condition also satisfies the economic requirement for "shadow prices" in that they make the chosen activities profitless, while all others incur losses in the sense of opportunity costs.[9]

The implications of this model for decentralization of investment decisions within the firm are not clear. Assuming no alternate optima, if the ρ_t^* were known and announced, each originator of a capital project would be able to judge whether his proposal is acceptable or not. If the outlays for a unit of investment, as evaluated by the ρ_t^*, are greater than the present worth b_j, then the proposal is unacceptable. The ρ_t^* perform an analogous role for the acceptable proposals. For these, the outlays, similarly evaluated, equal the present worth.

Acceptance of investment proposals, however, does not supply all the information required. The assumption of constant returns to scale implies the need to determine the scale of each of the accepted investments, and the dual prices, ρ_t^*, are insufficient for this purpose. For example, if, for some accepted proposal k

$$b_k - \sum_t \rho_t^* c_{tk} = 0 \qquad (4.6)$$

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[8]Strict inequality holds unless alternate optima are present in the primal solution.

[9]See, for example, R. Dorfman, P. A. Samuelson, and R. Solow, *Linear Programming and Economic Analysis* (New York: McGraw-Hill, 1958), Chapter 12, esp. pp. 339–40, and Chapter 14.

then also

$$b_k x_k - \sum_t \rho_t^* (c_{tk} x_k) = 0 \tag{4.7}$$

for all x_k. That is, if $j \in J$ represents the entire set of indices covering projects to be accepted, then the scale, x_j^*, of each investment in the accepted set J must still be determined from the conditions imposed by the budgets,

$$\sum_{j \in J} c_{tj} x_j^* \leq C_t.^{10} \tag{4.8}$$

As in our earlier models, the ρ_t^* are interest rates which evaluate use of scarce budget dollars. However, because the optimality condition (4.5) is an equation for each of the accepted proposals, all the projects are equally worthwhile with respect to their use of the limited funds. Hence the ρ_t^* do not supply information for ranking the proposals by the extent to which they use these scarce funds. Their role is rather to evaluate the marginal worth of dollars added to or subtracted from the C_t in each period $t = 1, \ldots, T$.

Similar remarks apply to the rejected projects. Thus the quantity

$$b_j - \sum_t \rho_t^* c_{tj} \tag{4.9}$$

cannot be used to rank the submarginal projects in certain important cases. For if, for some reason, a previously accepted proposal should become unavailable, the quantities (4.9), as computed from the available ρ_t^*, offer no guide to the efficient use of the budget dollars that are released by the proposal being withdrawn. The problem must be solved again, and, except by coincidence, a new set of ρ_t^* will then prevail.

It should be noted, in particular, that when one or more members of a previously accepted set are made irrelevant because, for whatever reason, they are no longer available, then it is not even possible to assert that the remaining proposals previously accepted will still be chosen. This observation may be demonstrated by means of the two period example portrayed in Fig. 4.2 for which the dual problem is solved graphically. Thus, by directly substituting in the formulas as given in the box of Fig. 4.2 we find that the constraints for the respective projects

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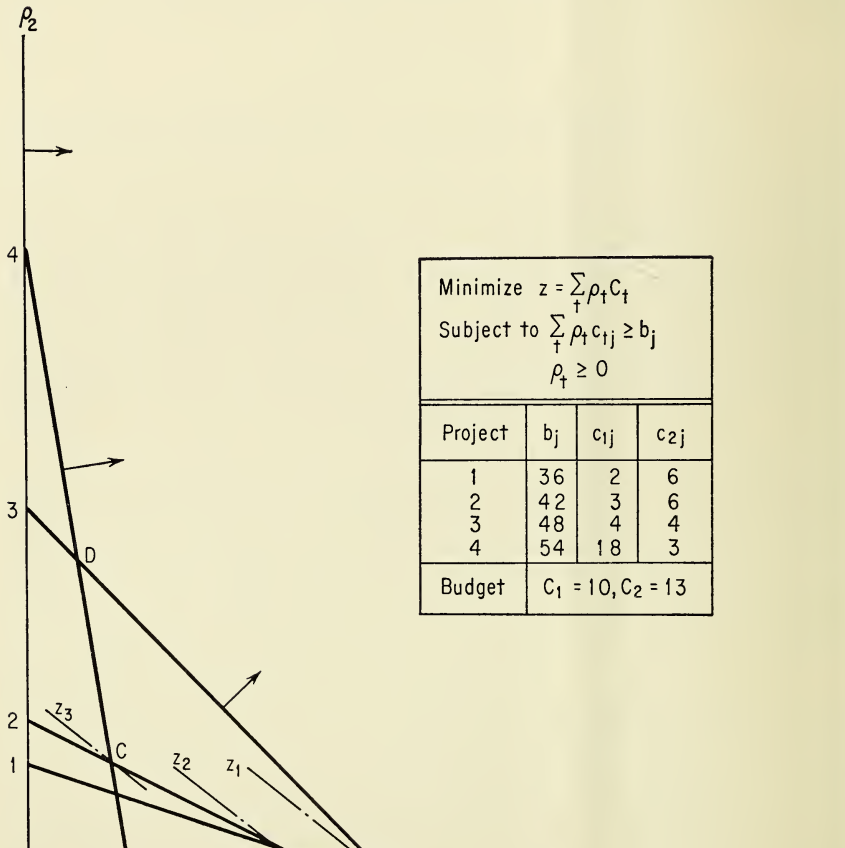
[10]It is still true, of course, that the optimal solution to the dual problem, including *its* dual evaluators, yields all the information required. However, this is not the same as the mere statement of the values of ρ_t^* since the indicated additional information on the dual evaluators (to the dual) designates, of course, the exact investment alternatives that are to be undertaken.

are

$$\begin{aligned} 1: &\quad 2\rho_1 + 6\rho_2 \geq 36 \\ 2: &\quad 3\rho_1 + 6\rho_2 \geq 42 \\ 3: &\quad 4\rho_1 + 4\rho_2 \geq 48 \\ 4: &\quad 18\rho_1 + 3\rho_2 \geq 54. \end{aligned} \tag{4.10}$$



Minimize $z = \sum_t \rho_t C_t$

Subject to $\sum_t \rho_t c_{tj} \geq b_j$

$\rho_t \geq 0$

Project	b_j	c_{1j}	c_{2j}
1	36	2	6
2	42	3	6
3	48	4	4
4	54	18	3
Budget	$C_1 = 10, C_2 = 13$		

Figure 4.2.

These are to be jointly satisfied with nonnegative ρ_1, ρ_2 which also minimize the functional $z = 10\rho_1 + 13\rho_2$.

Of the four investment alternatives in the example, projects 1 and 3 are chosen in the optimal solution. This is indicated by point A (with coordinates $\rho_1 = 9$, $\rho_2 = 3$) in the figure, which falls at the intersection of the lines whose equations are

$$2\rho_1 + 6\rho_2 = 36$$

and

$$4\rho_1 + 4\rho_2 = 48,$$

respectively. At this point the functional takes on its lowest value subject to being *on or above* the lines for all investments. See constraints (4.10).

If, say, the third proposal were to become unavailable so that its corresponding constraint would have to be eliminated from set (4.10) along with its bounding line and admissible half-space in Fig. 4.2, the optimal functional would shift from z_1 to z_3, and the optimum would shift from point A to point C, that is, to the intersection of the lines for alternatives 2 and 4 which are now the optimal proposals. Point B, the intersection of the lines for the first and second investments, falls on z_2 and above z_3, and thus the first proposal is now rejected although it is still available.

4.4. The Linear Programming Model with Upper Bounds

Next we take another look at the linear programming model with unity upper bounds on investments in order to analyze those aspects of the model which help to illuminate the difficulties that arise when the integer conditions are imposed. We focus our attention here on an economic interpretation of the dual variables of the model with upper bounds and on the role these quantities play as indices for ranking projects.

To facilitate our discussion we repeat here the primal and dual models of the Lorie-Savage problem presented earlier in systems (3.1) and (3.2) above.[11]

Primal

$$\begin{aligned} &\text{Maximize} && \sum_j b_j x_j \\ &\text{Subject to} && \text{(a)} \quad \sum_j c_{tj} x_j \leq C_t, \qquad t = 1, \ldots, T \qquad (4.11) \\ &&& \text{(b)} \quad x_j \leq 1, \\ &&& \text{(c)} \quad x_j \geq 0. \end{aligned}$$

[11]See pp. 17 and 24.

Dual

$$\text{Minimize} \qquad \sum_t \rho_t C_t + \sum_j \mu_j$$

$$\text{Subject to (a)} \quad \sum_t \rho_t c_{tj} + \mu_j \geq b_j, \qquad j = 1, \ldots, n \qquad (4.12)$$

$$\text{(b)} \quad \rho_t, \mu_j \geq 0.$$

Addition of the unity upper bounds on investments introduces variables μ_j into the dual problem. Imposition of the upper bounds increases the maximum number of nonzero projects substantially to include now the possibility of making all $x_j^* > 0$. As our earlier discussion of the dual variables of this model in Section 3.4 brought out, values of μ_j^* for the accepted projects are nonnegative while these quantities are zero for the rejected ones. That is,

$$\sum_t \rho_t^* c_{tj} \leq b_j$$

for the accepted projects, and

$$\sum_t \rho_t^* c_{tj} \geq b_j$$

for the rejected ones. These inequalities may be contrasted to the results in the preceding section in which investments without the unity upper bounds yielded equations (4.5) for the selected activities and strict inequalities for those rejected by the optimal program.[12] We see that it is no longer the case that the rates of interest, ρ_t^*, associated with the limited budget dollars are high enough to make the chosen activities "profitless."[13] This change derives from the presence of fixed factors — the upper limits to the x_j.

The return to fixed factors is customarily called economic rent. Here we have the discounted value of a stream of rents, which may be called the goodwill associated with this stream.[14] The μ_j^* are precisely the goodwill earned by the accepted projects that cannot be expanded beyond $x_j^* = 1$ where

$$\mu_j^* = b_j - \sum_t \rho_t^* c_{tj}. \qquad (4.13)$$

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[12]Again aside from possible alternate optima.

[13]In the sense of T. C. Koopmans and M. Beckmann, "Assignment Problems and Location," *Econometrica*, XXV, No. 1 (January, 1957), p. 59.

[14]See Friedrich and Vera Lutz, *The Theory of Investment of the Firm* (Princeton: Princeton University Press, 1951), p. 12.

One may similarly view the γ_j^* as "negative goodwill" associated with the rejected projects where the γ_j^* (the "$z_j^* - c_j^*$" of the primal problem) are defined, as in equation (3.4), by

$$\gamma_j^* = \sum_i \rho_i^* c_{ij} - b_j, \tag{4.14}$$

where $\gamma_j^* \geq 0$ is a condition of optimality.

These quantities, μ_j^* and γ_j^*, have properties that are useful for certain types of marginal analysis in that decreasing some x_j^* from the accepted set implies a reduction in total present value which will be proportional to μ_j^*; conversely, forcing an increase of some x_j^* from the rejected set implies a loss proportional to γ_j^*. Thus the μ_j^* and γ_j^* provide convenient indices for ranking projects.

One purpose for which rankings are typically established in capital budgeting procedures is to facilitate alterations in plans when unexpected events occur. For example, if some accepted project were to become unavailable, the highest ranking unselected project would generally be substituted. The presence of fixed budgets complicates the procedure in such a contingency. We may note, for instance, that fractional projects (if any) will have both $\mu_j^* = 0$ and $\gamma_j^* = 0$, that is, these earn zero goodwill in the previously optimal program. Thus it may be possible to allocate the funds released by the project that was withdrawn[15] among some of these fractional projects in order to expand them to unity.[16]

Despite the appearance of economic rent or goodwill, decentralization is still possible in the following sense. If the rates of interest, ρ_i^*, are announced, then each originator of a project may decide by himself whether or not to go ahead with his project. If the project earns a positive goodwill, that is, the right side of equation (4.13) is positive, then it is acceptable. If this quantity is negative, it should be rejected. In a no-goodwill-no-loss instance appeal to a central authority is necessary. Among these may be projects which are fractional in the optimal primal solution, and hence it will be necessary either to eliminate them or to adjust the budgets, if possible, to permit the inclusion of some of them *in toto*. It is also possible that some of these projects are totally acceptable, or corresponding to alternate optima in the primal problem, some of them may be totally unacceptable. Aside from these "borderline

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[15]As well as any unutilized funds of the budget ceilings, if there are any.

[16]It is also possible to employ parametric programming (see Section 7.4, pp. 129 ff., to obtain alternative programs under altered circumstances in computationally efficient ways.

cases," the dual prices permit decentralization because the additional scale factor, which had to be determined separately in the preceding model, is no longer required when unity upper bounds are placed on the investments.

The remarks above are all that we can — and need to — make regarding the dual variables and pricing within the context of our strict linear programming model. They are nevertheless indicative of the problems that arise when the integer constraints are also applied. It is to these problems, and particularly to the interpretation of the dual variables of integer programming problems, that we now must turn.

CHAPTER 5

The Integer Programming Algorithm and Imputation of the Dual Variables[1]

5.1. The Integer Programming Algorithm

The problem of obtaining the dual variables for integer programming problems is considered by R. E. Gomory and W. J. Baumol in an article[2] in which they also discuss the interpretation of these quantities. The chief difficulty is that the linear programming problem which is solved — the augmented problem — is not the problem which it is desired to solve. It is only an equivalent problem in the restricted sense that its optimum, in integers, also satisfies the optimality requirements of the problem that is really of interest. As already mentioned, we generally do not even know how to state the problem we wish to solve, *ab initio*, in an effective manner and so we proceed, instead, to solve a series of systematically augmented problems until one is achieved whose solution has the wanted integer properties.

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[1]This chapter is a digression from the main subject of capital budgeting and may be skipped without loss of understanding of subsequent material. The author wishes to acknowledge helpful discussions with Ralph Gomory regarding several sections of this chapter but absolves him of blame for errors and ambiguities which remain.

[2]R. E. Gomory and W. J. Baumol, "Integer Programming and Pricing," *Econometrica*, XXVIII, No. 3 (July, 1960), pp. 521–50.

To illustrate these remarks, and also to facilitate our later discussion of the process of imputation of the dual variables, we undertake first to state the algorithm and solve a simple numerical example of the Lorie-Savage type by means of Gomory's Method of Integer Forms.[3] In order to make easier comparisons with the Gomory and Baumol treatment of imputation, we adopt the notation used by Gomory; that is, we employ the condensed Simplex format of A. W. Tucker. In this formulation the identity matrix is omitted and the basic variables are expressed in terms of the nonbasic variables.

It may be simplest if we borrow a page from Gomory[4] in which he summarizes the Primal and Dual Simplex methods.[5] In his notation the functional is designated the zero row and the constants of the restrictions are the zero column.

> The simplex rule for choice of pivot element in both primal and dual methods can be summarized as follows: if the problem is primal [dual] feasible, that is, $a_{i,0} \geq 0$, $i \geq 1$ [$a_{0,j} \geq 0$, $j \geq 1$], choose a column j_0 [a row i_0] with first element a_{0,j_0} [$a_{i_0,0}$] negative. From among the positive [negative] elements in this column [row] select the one for which the ratio $\frac{a_{i,0}}{a_{i,j_0}}$ $\left[\frac{a_{0,j}}{a_{i_0,j}}\right]$ attains its least absolute value. This is the pivot element a_{i_0,j_0}.
>
> The effect of pivoting on pivot element a_{i_0,j_0} can be summarized as follows: a new array is obtained in which the variables at the end of the i_0 row and top of the j_0 column have been exchanged and in which the new coefficients $a'_{i,j}$ are given by

$$a'_{i,j} = \frac{a_{i,j} - a_{i_0,j}a_{i,j_0}}{a_{i_0,j_0}} \qquad i \neq i_0,\ j \neq j_0$$

$$a'_{i_0,j} = \frac{a_{i_0,j}}{a_{i_0,j_0}} \qquad j \neq j_0$$

$$a'_{i,j_0} = \frac{-a_{i,j_0}}{a_{i_0,j_0}} \qquad i \neq i_0$$

$$a'_{i_0,j_0} = \frac{1}{a_{i_0,j_0}}.$$

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[3] R. E. Gomory, "An Algorithm for Integer Solutions to Linear Programs," *op. cit.*

[4] *Ibid.*, pp. 46–47.

[5] That is, Lemke's Dual Method; see C. E. Lemke, "The Dual Method of Solving the Linear Programming Problem," *Naval Research Logistics Quarterly*, I, No. 1 (1954), pp. 36–47.

To obtain the integer optimum once the linear programming optimum has been achieved, new constraints are added as follows.

> When by a succession of pivots an array is obtained which is both primal and dual feasible (and hence optimal), a new equation (or equations) representing a new inequality is added. This equation is simply
>
> $$s = -f_0 - \sum_{j=1}^{j=n} f_j(-t_j)$$
>
> where the f_j are the fractional parts from some row and the t_j are the current nonbasic variables, or are the fractional parts obtained by combining (modulo 1) several of the fractional part rows, or multiplying a row by an integer. Here by fractional part f of an element a we mean the number obtained if an element a is written as $n + f$ with n an integer and $0 \leq f < 1$.
>
> After such an equation is added, the array is not primal feasible, so further steps of the Simplex method (usually the Dual method) are made until the array is again optimal. This process is then repeated until an integer answer (integer column of constants) or an all integer matrix (whichever is desired) is obtained.[6]

To illustrate this procedure we solve the following numerical example.

Problem 1

$$\begin{aligned} &\text{Maximize} && 3x_1 + 2x_2 \\ &\text{Subject to} && \text{(a)} \quad 4x_1 + 3x_2 \leq 5 \\ &&& \text{(b)} \quad x_1 \leq 1 \\ &&& \text{(c)} \quad x_2 \leq 1. \end{aligned} \tag{5.1}$$

The solution to this problem is obtained in Table 5.1 as Problem 1a.

In Tableau 1, the starting array, the functional to be maximized is stated in the first row in terms of the constant 0 and the coefficients which multiply the variables are written at the top. The barred variable, $\bar{x}_i$, represents the slack variable of the ith restriction. Thus the first restriction (5.1a) is made into an equation,

$$4x_1 + 3x_2 + \bar{x}_1 = 5, \tag{5.2}$$

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[6] *Ibid.* It is important that lexicographical ordering is followed for the resolution of primal and dual degeneracy.

TABLE 5.1. PROBLEM 1a AND SOLUTION

Maximize $3x_1 + 2x_2$

Subject to $4x_1 + 3x_2 \leq 5$

$x_1 \leq 1$

$x_2 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =	0	−3	−2
$\bar{x}_1$ =	5	4	3
$\bar{x}_2$ =	1	1*	
$\bar{x}_3$ =	1		1

Tableau 2

	1	$-\bar{x}_2$	$-x_2$
z =	3	3	−2
$\bar{x}_1$ =	1	−4	3*
x_1 =	1	1	
$\bar{x}_3$ =	1		1

Tableau 3

	1	$-\bar{x}_2$	$-\bar{x}_1$
z =	11/3	1/3	2/3
→ x_2 =	1/3	−4/3	1/3
x_1 =	1	1	
$\bar{x}_3$ =	2/3	4/3	−1/3
s_1 =	−1/3	−2/3*	−1/3

(L.P. optimum)

Tableau 4

	1	$-s_1$	$-\bar{x}_1$
z =	7/2	1/2	1/2
x_2 =	1	−2	1
→ x_1 =	1/2	3/2	−1/2
$\bar{x}_3$ =		2	−1
$\bar{x}_2$ =	1/2	−3/2	1/2
s_2 =	−1/2	−1/2	−1/2*

Tableau 5

	1	$-s_1$	$-s_2$
z =	3		1
x_2 =		−3	2
x_1 =	1	2	−1
$\bar{x}_3$ =	1	3	−2
$\bar{x}_2$ =		−2	1
$\bar{x}_1$ =	1	1	−2

(Integer optimum)

which may be transposed to express the slack variable in terms of the negatives of the original variables,

$$\bar{x}_1 = 5 + 4(-x_1) + 3(-x_2). \tag{5.3}$$

Similarly, the upper bound constraints on x_1 and x_2 are changed into equations by addition of slack variables and entered into the tableau.

The initial array is primal feasible and the pivot element, which is marked by an asterisk, is identified by the rules quoted above. That is, the column headed by $-x_1$ has a negative element in the zero row, −3,

and the starred element, 1, in that column and the row of the second constraint minimizes the ratio $\frac{a_{i,0}}{a_{i,j_0}}$ for elements $a_{i,j_0} > 0$: $\frac{1}{1} < \frac{5}{4}$.

Tableaux 2 and 3 are similarly obtained by performing the Simplex algorithm. The elements of the zero row and the zero column of Tableau 3 are all nonnegative. Hence, this solution is primal and dual feasible, and therefore the optimal linear programming solution has been obtained: $x_1 = 1$, $x_2 = \frac{1}{3}$, $\bar{x}_3 = \frac{2}{3}$ and $z = \frac{11}{3}$. The solution is seen to be fractional, and so we use the row marked with an arrow to form the cutting plane

$$\frac{2}{3}\bar{x}_2 + \frac{1}{3}\bar{x}_1 \geq \frac{1}{3} \tag{5.4}$$

which may be entered into the tableau with the aid of the slack variable s_1 for this added restriction, namely,

$$s_1 = -\frac{1}{3} - \frac{2}{3}(-\bar{x}_2) - \frac{1}{3}(-\bar{x}_1) \tag{5.5}$$

in accordance with the quoted rule for generating cutting planes.[7] The solution to this new problem, that is, the previous linear programming optimum but with the added constraint, is not primal feasible, hence the Dual method is employed to obtain the solution exhibited in Tableau 4. The slack vector of the added constraint is made nonbasic and the original slack $\bar{x}_2$ enters the basis.

The solution of Tableau 4 is again optimal since it is both primal and dual feasible, but it is again fractional, and so another cutting plane must be employed. This time it is derived from the equation for x_1, the row marked by an arrow. The new restriction is given the slack variable s_2 and is entered into the tableau below the line. The restriction, in terms of the variables of the tableau, is

$$\frac{1}{2}s_1 + \frac{1}{2}\bar{x}_1 \geq \frac{1}{2}. \tag{5.6}$$

Note that in this form the restriction has been stated in terms of a

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[7] In constructing the cutting plane from the row indicated by an arrow in Tableau 3, we determine the fractional part of the term in the zero column, that is, $f(\frac{1}{3}) = \frac{1}{3}$; the fractional part of the term in the column headed by $-\bar{x}_2$, that is, $f(-\frac{4}{3}) = \frac{2}{3}$, and the fractional part of the term in the column headed by $-\bar{x}_1$, that is, $f(\frac{1}{3}) = \frac{1}{3}$. The first of these quantities is f_0; the remaining two are the f_j associated with the currently nonbasic variables $\bar{x}_2$ and $\bar{x}_1$, respectively. Together they yield the cutting plane (5.5).

variable, s_1, which did not appear in the original problem. As we shall show in a moment, the added constraint can be stated as a function of the original basic and nonbasic variables.

Tableau 5 is obtained by removing the infeasibility introduced by the second cutting plane. This solution is again both primal and dual feasible, and this time it is also in integers.[8] Hence we have arrived at the optimal integer solution: $x_1^* = 1$, $\bar{x}_1^* = 1$, $\bar{x}_3^* = 1$, $x_2^* = 0$ and $\bar{x}_2^* = 0$. The value of the solution is $z^* = 3$.

We may illustrate the operation of the cutting planes by means of a geometric solution to this problem and utilize the example to show how the cutting planes are stated in terms of the variables of the original problem. Figure 5.1 presents the problem in terms of only two variables, x_1 and x_2. The original feasible region is given by the horizontally shaded region $OABCD$, bounded by the nonnegativity constraints on the horizontal and vertical axes; by the upper bound constraint on x_1, line AB; by the upper bound on x_2, line CD; and by the constraint (5.1a), line BC. The first linear programming solution of Tableau 2 is located at point A, and the optimal linear programming solution without the addition of new constraints is located at point B.

The first cutting plane, given by inequality (5.4) may be restated in terms of the original variables x_1 and x_2 by utilizing the equations defining the slack variables in Tableau 1. We observe, first, that the variables $\bar{x}_2$ and $\bar{x}_1$ are given by the following:

$$\bar{x}_2 = 1 - x_1 \tag{5.7}$$

and

$$\bar{x}_1 = 5 - 4x_1 - 3x_2. \tag{5.8}$$

Making these substitutions into (5.4) we obtain

$$\frac{2}{3}(1 - x_1) + \frac{1}{3}(5 - 4x_1 - 3x_2) \geq \frac{1}{3} \tag{5.9}$$

which simplifies to

$$2x_1 + x_2 \leq 2. \tag{5.10}$$

This restriction is represented in the figure by the dashed line through A and C and is seen to cut away the triangle ABC from the feasible region.

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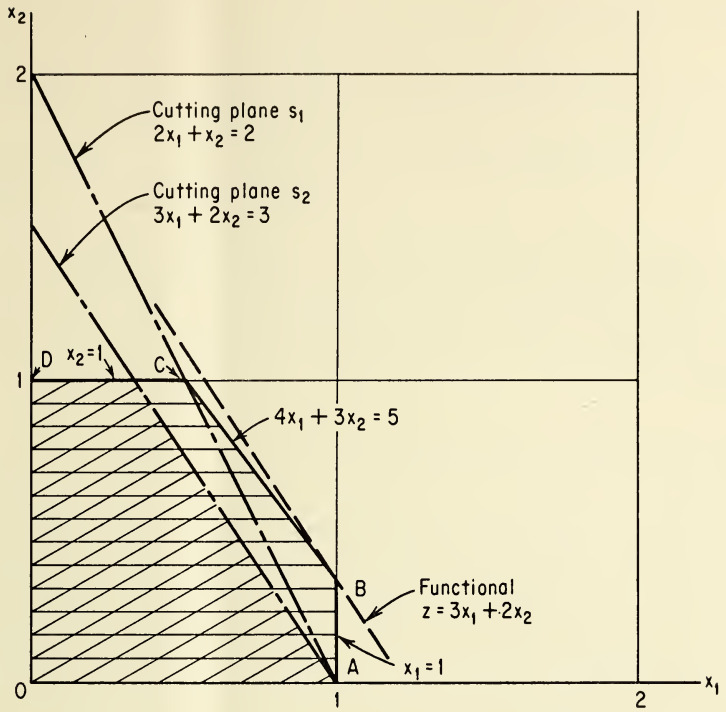
[8]It is not necessarily the case that removing the infeasibility immediately restores optimality. Rather, it may take several iterations to bring about a new linear programming optimum.



A = First Solution
B = Second Solution = L. P. Optimum
C = Third Solution — After first cutting plane
A = Fourth Solution = Integer Optimum

[The original feasible region, apart from the requirement of discreteness, is horizontally shaded. The final feasible region again neglecting discreteness is cross-hatched.]

Figure 5.1.

The second cutting plane, equation (5.6), is given in terms of $\bar{x}_1$ which is defined by (5.8) and s_1, which is the slack variable of inequality (5.10) and has already been given in terms of the original variables x_1 and x_2, namely,

$$s_1 = 2 - 2x_1 - x_2. \tag{5.11}$$

This enables us to restate the second cutting plane as

$$\frac{1}{2}(2 - 2x_1 - x_2) + \frac{1}{2}(5 - 4x_1 - 3x_2) \geq \frac{1}{2} \tag{5.12}$$

which simplifies to

$$3x_1 + 2x_2 \leq 3. \tag{5.13}$$

This cutting plane has also been plotted in Fig. 5.1 and is seen also to go through point A but to cut away point C from the feasible set. The new linear programming solution to the problem augmented by the second cut is at point A, which is integral, and hence it is the solution sought.

We note that the second cutting plane makes the first one redundant: the feasible region after imposition of the second Gomory constraint is the same whether the first cutting plane is in effect or not. However, in the computations this second additional constraint was derived from the first.[9]

5.2. Imputation of the Dual Variables

The problem of imputation of the dual variables arises in the integer programming problem from the additional constraints which have to be added to obtain the integer solution. The integer optimum was obtained by solving a linear programming problem which was the original problem augmented by a number of constraints. The linear programming problem with these additional constraints added will also have a dual, and the optimal solution to this dual problem will include values for the variables associated with the additional constraints in the primal problem. However, the new constraints ordinarily do not carry with them an interpretation based on the original sense of the problem. They are an expression of the indivisibility requirement, and the dual variables related to these may be thought of as "a measure of the opportunity cost of indivisibility — for example, the loss imposed on the businessman by a unit of the artificial capacity constraint which prevents him from seeking to stuff that last four-tenths of a case into his warehouse."[10] For reasons such as these it is desirable not to leave the problem there but rather to find quantities which evaluate the original constraints alone. This is the object of the Gomory and Baumol article.[11]

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[9]Alternative cutting planes could have been employed to solve this problem.

[10]Gomory and Baumol, *op. cit.*, p. 530.

[11]The present and several subsequent sections of this chapter will be utilized to restate, illustrate, and enlarge upon the process of recomputation as formulated by Gomory and Baumol. The arguments are chiefly derived from the appendices to their article. Unfortunately, these contain a number of typographical errors, and the tightness of their arguments makes interpretation difficult, especially in the absence of numerical examples of recomputing.

The Gomory-Baumol method of imputation of shadow prices to the original constraints is intimately tied to the Method of Integer Forms just discussed.[12] As we have observed, this algorithm utilizes linear combinations of prior constraints to form the additional restrictions, and this makes it possible to reverse the process in order to impute the dual evaluators associated with the added constraints back to the original restrictions from which they were derived.

The process of "recomputing" of Gomory and Baumol works iteratively, retracing backwards the steps taken to arrive at the integer optimum. Initially, the dual price associated with the last cutting plane added is distributed to the constraints from which it was generated. Since a previous cutting plane may have been one of the constraints from which the final cutting plane was formed, a portion of the dual price of the final cutting plane may be distributed back to this earlier one. Once the distribution of the final cutting plane has been accomplished, the dual price of the immediately prior cutting plane is distributed back to the constraints utilized to generate it. This process is continued until the duals of all added constraints have been distributed back to the original constraints. At this point the process is completed since only dual evaluators of the original constraints remain.

Following the notation of Gomory and Baumol, we denote the final dual values of the integer programming solution by π_i, where the subscript denotes the number of the constraint with which it is associated. Specifically, the first m constraints are the original constraints of the problem it is desired to solve. Constraints $m + 1$ to $m + n$ are the n nonnegativity constraints $-x_j \leq 0$ which are considered appended to the constraint set. Finally, constraints $m + n + 1$ to $m + n + q$ are the constraints which are added during the course of the computations, the cutting planes of the form

$$s_r = -f_{r,0} + \sum_J f_{r,j} t_j,$$

where, as before, the $f_{r,j}$ are the fractional parts of some row, and the t_j are the currently nonbasic variables comprising the set J. This set may, of course, include slack variables of some earlier cutting planes. For convenience we shall single out from among the π_i the dual variable

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[12]The problem of imputation when Gomory's All-Integer Integer Programming Algorithm is used has not been discussed. The solution to Problem 1a using this algorithm (without imputation) is presented in Appendix 5A as Problem 1b.

associated with the last added cutting plane by calling it π_p. We shall also mark a revised dual by a prime.[13]

In the first step the dual price π'_p is set to zero and the other values of π_i are revised as follows:

$$\pi'_i = \pi_i + f_{pi}\pi_p. \tag{5.14}$$

From its origin in the optimal solution we know that π_p must be nonnegative. Similarly, by construction the f_{pi} are also nonnegative. Hence the revised dual price π'_i is increased if it is changed. After the first revision, if a dual price associated with the slack of a cutting plane is greater than zero, the process must be repeated. When all positive duals associated with the added constraints have been distributed back to original constraints the process is ended.

We illustrate the procedure by reference to Problem 1a whose solution was given in Table 5.1. In the integer optimum of Tableau 5 only the dual associated with the second cutting plane has a positive value. This is the value 1 in the zero row in the s_2 column. The value of the dual associated with s_1 in the final tableau is zero. If we denote the dual prices associated with the original three constraints by π_1, π_2, and π_3, respectively, and further associate π_4 and π_5 with the nonnegativity constraints $-x_1 \leq 0$, $-x_2 \leq 0$, then we shall be consistent in denoting the dual price of the first cutting plane by π_6 and that of the second cutting plane by π_7. That is, in the integer optimal solution, $\pi_6 = 0$, $\pi_7 = 1$.

The first step in the recomputation involves revising the dual prices associated with the constraints utilized in stating the second cutting plane. In Tableau 4 this was given by

$$s_2 = -\frac{1}{2} + \frac{1}{2}s_1 + \frac{1}{2}\bar{x}_1. \tag{5.15}$$

---

[13]In common with usual linear programming notation, we presume the values of all π_i not appearing in the optimal tableau to be zero. We are here at variance with the notation of Gomory and Baumol, in which the π_i are "efficiency prices" of activity analysis. [See T. C. Koopmans (ed.), "Analysis of Production as an Efficient Combination of Activities," *Activity Analysis of Production and Allocation* (New York: John Wiley & Sons, Inc., 1951) pp. 65, 96.] In activity analysis concern is over prices which sustain an optimal solution, the primal functional never appearing explicitly. Instead, in terms of Gomory and Baumol (*op. cit.*, p. 546) for $i = m + 1, \ldots, m + n$, $\pi_i = -a_{0,j}$. To establish correspondence between these π_i as used by Gomory and Baumol and as used by us we need only subtract $a_{0,j}$ from our quantities to obtain theirs. An indication for the necessity for this may be seen in the form of Gomory and Baumol's equations (B.1) and (B.2) (*op. cit.*, p. 546) which are derived from the activity analysis approach.

Accordingly we may distribute the value $\pi_7 = 1$ to π_6 and π_1 by use of (5.14):

$$\pi_6' = 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}, \tag{5.16}$$

and

$$\pi_1' = 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}. \tag{5.17}$$

(Since $\bar{x}_1$ appeared in the final basis, its dual, π_1, must necessarily have been 0.)

At this point the dual associated with the first cutting plane is positive, and another step is required. In Tableau 3 this was given as

$$s_1 = -\frac{1}{3} + \frac{2}{3}\bar{x}_2 + \frac{1}{3}\bar{x}_1. \tag{5.18}$$

Again utilizing relation (5.14) we obtain the second revision:

$$\pi_2'' = 0 + \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3} \tag{5.19a}$$

and

$$\pi_1'' = \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{2}{3}. \tag{5.19b}$$

In (5.19b), the first term of $\frac{1}{2}$ on the right side is the result of the previous revision, π_1'. Since no positive dual values associated with any added constraints remain, we may exhibit the recomputed dual values: $\pi_1^* = \frac{2}{3}$, $\pi_2^* = \frac{1}{3}$, $\pi_3^* = \pi_4^* = \pi_5^* = 0$. In addition, $\pi_6^* = \pi_7^* = 0$.

To facilitate display of the recomputation process, we shall henceforth utilize a tabular form in which the rows are identified by the slack variables of the constraints, and the columns give the revision step by step. For the nonnegativity constraints we may regard the variable as its own slack, namely, when $-x_j \leq 0$ is replaced by $\bar{x}_j - x_j = 0$, or $\bar{x}_j = x_j$. Thus we list all the slack variables, including those of the cutting planes. For Problem 1a, this takes the form of Table 5.2.

We shall return shortly to the question of interpreting these as shadow prices in the context of our original Lorie-Savage budgeting models. Before we do so, however, we find it useful to discuss these quantities more generally, and to review their properties.

TABLE 5.2. RECOMPUTED DUAL VALUES FOR PROBLEM 1a

	π	π'	$\pi'' = \pi^*$
$\bar{x}_1$	0	$0 + (1/2)1$	$(1/2) + (1/3)(1/2) = 2/3$
$\bar{x}_2$	0	0	$0 + (2/3)(1/2) = 1/3$
$\bar{x}_3$	0	0	$0 = 0$
x_1	0	0	$0 = 0$
x_2	0	0	$0 = 0$
s_1	0	$0 + (1/2)1$	—
s_2	1	—	—

5.3. Properties of the Recomputed Dual Variables

In this section we propose to illustrate the dependence of the recomputed duals on the choice of cutting planes employed in the integer optimization, and to compare these duals with their counterparts in ordinary linear programming, especially with regard to their role as shadow prices.

As we have just observed, to perform the recomputation requires a complete history of the computations. Specifically, an explicit statement of all added constraints is needed, even though some of these constraints may be redundant in the end, and even though these constraints are of no interest, except for recomputation, once the integer optimum has been obtained.[14]

The dependence of the values of the recomputed duals on the choice of cutting planes may be illustrated by means of a second problem which has been solved in two different ways. In the first of these, Problem 2a of Table 5.3, a cutting plane is derived from the second row of the second tableau, the linear programming optimum. Its equation is

$$s_1 = -\frac{1}{2} - \frac{1}{3}(-x_1) - \frac{1}{6}(-\bar{x}_1).$$

As the computations of the table show, no further cutting planes are needed to obtain the integer optimum. The recomputation at the end of the table is performed in one step, distributing the final dual of the added constraint, $\pi_6 = 8$, to the original constraints.

Table 5.4 presents the same problem as Problem 2b in which different cutting planes are employed. The first cutting plane, imposed in Tableau 2,

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[14]Thus, for example, when an alternative algorithm is used, such as the All-Integer algorithm referred to earlier, it may not be possible to obtain the recomputed duals by these methods.

TABLE 5.3. PROBLEM 2a AND SOLUTION

Maximize $2x_1 + 8x_2$
Subject to $2x_1 + 6x_2 \leq 3$
$x_1 \leq 1, \quad x_2 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−2	−8
$\bar{x}_1$ =	3	2	6*
$\bar{x}_2$ =	1	1	
$\bar{x}_3$ =	1		1

Tableau 2

	1	$-x_1$	$-\bar{x}_1$
z =	4	2/3	8/6
→ x_2 =	1/2	1/3	1/6
$\bar{x}_2$ =	1	1	
$\bar{x}_3$ =	1/2	−1/3	−1/6
s_1 =	−1/2	−1/3*	−1/6

(L.P. optimum)

Tableau 3

	1	$-s_1$	$-\bar{x}_1$
z =	3	2	1
x_2 =		1	
$\bar{x}_2$ =	−1/2	3	−1/2*
$\bar{x}_3$ =	1	−1	
x_1 =	3/2	3	1/2

Tableau 4

	1	$-s_1$	$-\bar{x}_2$
z =	2	8	2
x_2 =		1	
$\bar{x}_1$ =	1	−6	2
$\bar{x}_3$ =	1	−1	
x_1 =	1	6	1

(Integer optimum)

Recomputation

	π	π^*
$\bar{x}_1$	0	$0 + (1/6)8 = 4/3$
$\bar{x}_2$	2	$2 = 2$
$\bar{x}_3$	0	$0 = 0$
x_1	0	$0 + (1/3)8 = 8/3$
x_2	0	$0 = 0$
s_1	8	—

the linear programming optimum, is generated from the fourth row of that tableau. Its equation is

$$s_1 = -\frac{1}{2} - \frac{2}{3}(-x_1) - \frac{5}{6}(-\bar{x}_1).$$

The result of pivoting in the row of the new constraint is a feasible but noninteger solution, Tableau 3, which calls for a new cutting plane. This

TABLE 5.4. PROBLEM 2b AND SOLUTION

Maximize $2x_1 + 8x_2$
Subject to $2x_1 + 6x_2 \leq 3$
$x_1 \leq 1, \quad x_2 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−2	−8
$\bar{x}_1$ =	3	2	6*
$\bar{x}_2$ =	1	1	
$\bar{x}_3$ =	1		1

Tableau 2

	1	$-x_1$	$-\bar{x}_1$
z =	4	2/3	8/6
x_2 =	1/2	1/3	1/6
$\bar{x}_2$ =	1	1	
→ $\bar{x}_3$ =	1/2	−1/3	−1/6
s_1 =	−1/2	−2/3*	−5/6

(L.P. optimum)

Tableau 3

	1	$-s_1$	$-\bar{x}_1$
z =	7/2	1	1/2
x_2 =	1/4	1/2	−1/4
$\bar{x}_2$ =	1/4	3/2	−5/4
→ $\bar{x}_3$ =	3/4	−1/2	1/4
x_1 =	3/4	−3/2	5/4
s_2 =	−3/4	−1/2	−1/4*

Tableau 4

	1	$-s_1$	$-s_2$
z =	2		2
x_2 =	1	1	−1
$\bar{x}_2$ =	4	4	−5
$\bar{x}_3$ =		−1	1
x_1 =	−3	−4*	5
$\bar{x}_1$ =	3	2	−4

Tableau 5

	1	$-x_1$	$-s_2$
z =	2		2
→ x_2 =	1/4	1/4	1/4
$\bar{x}_2$ =	1	1	
$\bar{x}_3$ =	3/4	−1/4	−1/4
s_1 =	3/4	−1/4	−5/4
$\bar{x}_1$ =	3/2	1/2	−3/2
s_3 =	−1/4	−1/4*	−1/4

Tableau 6

	1	$-s_3$	$-s_2$
z =	2		2
x_2 =		1	
$\bar{x}_2$ =		4	−1
$\bar{x}_3$ =	1	−1	
s_1 =	1	−1	−1
$\bar{x}_1$ =	1	2	−2
x_1 =	1	−4	1

(Integer optimum)

Recomputation

	π	π'	π^*
$\bar{x}_1$	0	0 + (1/4)2	1/2 + (5/6)1 = 4/3
$\bar{x}_2$	0	0	0 = 0
$\bar{x}_3$	0	0	0 = 0
x_1	0	0	0 + (2/3)1 = 2/3
x_2	0	0	0 = 0
s_1	0	0 + (1/2)2	——
s_2	2	——	——
s_3	0	——	——

additional constraint is

$$s_2 = -\frac{3}{4} - \frac{1}{2}(-s_1) - \frac{1}{4}(-\bar{x}_1)$$

and is obtained from the row indicated by the arrow.[15]

A third cutting plane is required in this solution. When primal feasibility is restored in Tableau 5 of Table 5.4, the solution is still noninteger. The third and final cutting plane

$$s_3 = -\frac{1}{4} - \frac{1}{4}(-x_1) - \frac{1}{4}(-s_2)$$

is derived from the constraint indicated once again by the arrow. This now yields the integer optimum.

Recomputation, as shown at the bottom of Table 5.4, requires only two steps. The final dual value associated with the third cutting plane is zero, and hence nothing from it must be distributed to earlier constraints.

When the values of the recomputed duals at the end of Tables 5.3 and 5.4 are compared, as in Table 5.5, they are found to be different. Not

TABLE 5.5. RECOMPUTED DUAL VALUES FOR PROBLEMS 2a AND 2b

	Problem 2a	Problem 2b
π_1^*	4/3	4/3
π_2^*	2	0
π_3^*	0	0
π_4^*	8/3	2/3
π_5^*	0	0

only are the numerical values different, in part, but they also assign positive values to different constraints.

The recomputed duals, in common with ordinary linear programming dual variables, have the property that they are "sufficiently high to eliminate the possibility of any profitable output, and an output will be

[15]Other cutting planes could have been obtained by utilizing rows three or four of Tableau 3, or by using a convex combination of the rows of that tableau. It would also have been legitimate to utilize the functional row for generating a cutting plane. In the Method of Integer Forms it is required that the coefficients of the functional be integers. This implies that the value of the integer solution, if it exists, is also integral. Hence it is possible to form cutting planes from the zero row as well as from other rows of the tableaux.

produced if and only if it yields zero profits."[16] In the context of the Lorie-Savage problem, this implies that for the set J of projects accepted in the *optimal integer* solution,

$$\sum_t \rho_t^* c_{tj} + \mu_j^* = b_j, \qquad j \in J \tag{5.20}$$

and

$$\sum_t \rho_t^* c_{tk} + \mu_k^* \geq b_k, \qquad k \notin J. \tag{5.21}$$

We may illustrate this in the context of our earlier example, Problem 1a. In the notation we have adopted for the Lorie-Savage models the recomputed duals of equations (5.18) and (5.19) would be stated as

$$\rho_1^* = \frac{2}{3}, \qquad \mu_1^* = \frac{1}{3}, \qquad \mu_2^* = 0. \tag{5.22}$$

Since the only project accepted in Problem 1a is the first, that is, $x_1^* = 1$ and $x_2^* = 0$, we obtain from equation (5.20)

$$\rho_1^* c_{11} + \mu_1^* = b_1, \tag{5.23}$$

or, substituting the numerical values above,

$$\left(\frac{2}{3}\right)4 + \frac{1}{3} = 3. \tag{5.24}$$

Similarly, from equation (5.21) we obtain

$$\rho_1^* c_{12} + \mu_2^* \geq b_2. \tag{5.25}$$

Numerically, this comes to the following for the above example,

$$\left(\frac{2}{3}\right)3 + 0 = 2; \tag{5.26}$$

that is, equality holds in this instance.

Interpretation of these numerical values as evaluators of the relevant constraints is more difficult. For example, it is not clear what the value $\rho^* = \frac{2}{3}$ means. If the right side of the budget constraint is reduced by 1 the optimum does not change since $\bar{x}_1^* = 1$; that is, the slack variable is in the optimal basis at a nonzero level. If the right side term were reduced by an amount sufficient to affect the solution, as, for example, if its value were 3 instead of the present 5, the resulting reduction in the program value would be \$1. The new solution would be $x_1^* = 0$, $x_2^* = 1$,

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[16] Gomory and Baumol, *op. cit.*, p. 533.

$z^* = 2$. Thus a reduction in the budget of \$2 would mean a reduction in the solution value of \$1, a ratio of $\frac{1}{2}$ and not $\frac{2}{3}$. Neither does relaxation of this constraint explain the numerical value obtained for the dual. An increase in the budget of \$1 would not affect the integer optimum, and an increase of \$2 would result in a new solution with $z^* = 5$, $x_1^* = 1$, $x_2^* = 1$. The ratio then is $\frac{2}{2} = 1$. We may thus conclude that the recomputed dual variables do not necessarily express the opportunity cost associated with discrete changes in the right side of a given constraint.

An important departure from ordinary linear programming is the following. The recomputed duals do not necessarily satisfy the dual theorem. In general,

$$z^* = \sum_{j=1}^{n} a_{0,j} x_j^* \neq \sum_{i=1}^{m} \pi_i^* a_{i,0} \tag{5.27}$$

where the $a_{0,j}$ are the coefficients of x_j in the functional, the π_i^* are the recomputed dual values and the $a_{i,0}$ are the values of the original right side.[17] Thus they do not "cost" the constraints in a way which is consistent with the optimization itself.

The numerical values obtained for the duals of Problem 1a may be used to illustrate relation (5.27). The value of the integer optimum, as found in Table 5.1, is $z^* = 3$. Using dual prices (5.22) to value the original constraints, as in the right side of the inequality in (5.27), yields

$$5\pi_\rho^* + 1\mu_1^* + 1\mu_2^* = 5\left(\frac{2}{3}\right) + \frac{1}{3} + 0 = \frac{11}{3} \neq 3. \tag{5.28}$$

One aspect relating to the failure of the usual duality relation of linear programming in integer programming is the following property of the recomputed duals. In order to carry out the process of recomputation it may be necessary to impute positive values to some of the original nonnegativity constraints. "Among the inequalities which make up the artificial constraints there may be included some of the final output nonnegativity conditions, $x_i \geq 0$. It follows that *some of the artificial constraint prices may be reimputed, in part, to some of these final outputs.*"[18]

This situation, already encountered in Problem 2, may be analyzed more readily in terms of our earlier problem. In Problem 1a we note that restriction (5.1a), the "budget constraint", is sufficiently restrictive as to make the upper bounds of unity on each of the x_j unnecessary. A complete relaxation of either or both these restrictions would not alter the

[17] An obvious exception, in which equality does hold, occurs when the linear programming optimum solution is in integers, obviating the use of cutting planes.

[18] *Ibid.*, p. 533.

integer optimum solution. We have solved this problem without these redundant constraints as Problem 1c in Table 5.6. As the resulting recomputation at the end of the table shows, a positive dual price is now associated with the constraint $x_2 \geq 0$.

TABLE 5.6. PROBLEM 1c AND SOLUTION

Maximize $3x_1 + 2x_2$
Subject to $4x_1 + 3x_2 \leq 5$

Tableau 1

	1	$-x_1$	$-x_2$
$z =$		-3	-2
$\bar{x}_1 =$	5	4*	3

Tableau 2

	1	$-\bar{x}_1$	$-x_2$
$\rightarrow z =$	15/4	3/4	1/4
$x_1 =$	5/4	1/4	3/4
$s_1 =$	$-3/4$	$-3/4$*	$-1/4$

(L.P. optimum)

Tableau 3

	1	$-s_1$	$-x_2$
$z =$	3	1	
$x_1 =$	1	1/3	2/3
$\bar{x}_1 =$	1	$-4/3$	1/3

(Integer optimum)

Recomputation

	π	π^*
$\bar{x}_1$	0	$0 + (3/4)1 = 3/4$
x_1	0	$0 = 0$
x_2	0	$0 + (1/4)1 = 1/4$
s_1	1	——

How this arises geometrically may be seen from Fig. 5.1 in which this problem was graphed originally. When the upper bound constraints are neglected, the points $x_1 = 2$, $x_2 = -1$; $x_1 = 4$, $x_2 = -4$, etc., all lying in the fourth quadrant, that is, below the x_1 axis but above the line $3x_1 + 2x_2 = 3$, have a higher functional value than the optimum at $x_1 = 1$, $x_2 = 0$. Since the coefficient of x_1 in the functional is 3 and the coefficient of x_2 is 2, were it not for this nonnegativity constraint a trade-off of x_2 for x_1 would be desirable. When the upper bounds are added to the problem, such solutions are infeasible. Thus in the absence of these, the coordinate axis plays a distinct role as a binding restriction, and a dual price is imputed to it.

An economic interpretation of a positive dual price associated with a nonnegativity constraint suggests itself when this condition is explicitly appended to the constraint set of Problem 1c in the form

$$-x_2 \leq 0. \tag{5.29}$$

The corresponding dual constraint then becomes

$$3\rho - \gamma_2 \geq 2. \tag{5.30}$$

The positive quantity γ_2 may be regarded as evidencing a "negative value" or deficiency associated with the affected project in the sense of reversing inequality (5.21). Thus γ_2 is in the nature of a "subsidy" which is required to maintain the interpretation of the dual variables as "prices" which "eliminate the possibility of any profitable output," and which lead to production of an output "if and only if it yields zero profits."[19]

A most interesting additional property of the recomputed dual variables is the following.

> Any input with a zero recomputed price will be a free good in the true economic sense. That is to say, an unlimited increase in the stocks of this good will make absolutely no difference in optimum output levels.[20]

However, the converse of this statement does not hold. "Some free goods may not be given zero recomputed prices."[21] This may again be seen in Problem 1a in which the "unnecessary" upper bound constraint $x_1 \leq 1$ receives the recomputed value of $\frac{1}{3}$. As just demonstrated, omission of this constraint does not alter the optimum (given the implicit restriction $x_2 \geq 0$).

5.4. Nature of the Linear Programming Dual Evaluators

In linear programming, because of the implied infinite divisibility of all inputs and all outputs, the appearance of a zero dual variable is generally (but not always) associated with a nonbinding constraint in a way which suggests that the omission of the indicated constraint would not affect the optimal solution.[22] With indivisible resources, as we have here, this need not be the case. Specifically, an original constraint may not be binding at the integer optimum; that is, its slack may be in the final basis, and yet its omission would fail to yield the integer solution. Such a situation may be illustrated by means of the following diagram.

---

[19] *Ibid.*

[20] *Ibid.*

[21] *Ibid.*

[22] More precisely, the indicated constraint can be omitted if and only if its associated dual variable is zero in *every* optimal solution. This will be true if in *any* optimal solution the corresponding primal constraint is fulfilled as a *strict* inequality in any optimal solution. See, for example, A. J. Goldman and A. W. Tucker, "Theory of Linear Programming," *Linear Inequalities and Related Systems, op. cit.*, p. 62.

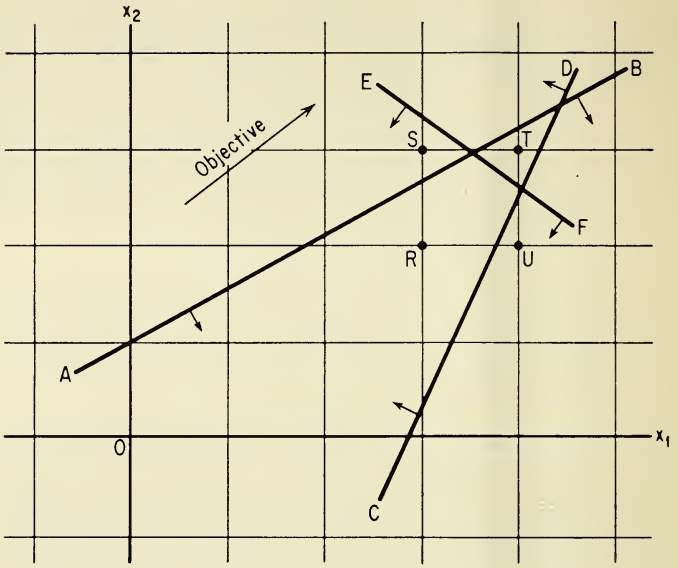

Figure 5.2.

In Fig. 5.2 the function to be maximized is considered to increase in the direction of the arrow. Aside from nonnegativity, the feasible region lies below the lines AB and EF, and to the left of line CD. Clearly, the optimal lattice point is point R. Even though the solutions-space is two dimensional, all three restrictions here are essential. If restriction EF is omitted, the optimum shifts to lattice point T; if the restriction AB is omitted, the optimum shifts to lattice point S; and if restriction CD is omitted, the optimum shifts to lattice point U.

Although none of the restrictions in Fig. 5.2 is actually binding at the integer optimum, cutting planes derived from all of them are necessary to achieve the optimum, point R. In linear programming a zero dual price is associated with a basic slack variable — a constraint with "excess capacity." In integer programming, even with the appearance of the slack of a constraint in the optimal basis at a positive level, there is no assurance that the restriction is not essential. When, however, its recomputed dual price is zero, it is deemed "redundant" — its corresponding capacity is regarded as a "free good." This interpretation depends heavily on the remaining restrictions of the problem. It may be that some constraint is unnecessary and hence is absolutely redundant in the following sense. If, in *all* subsets of constraints which yield the integer optimum by application of the algorithm, a given constraint

receives a recomputed dual price of zero, it is truly redundant. It may nevertheless be the case that a particular restriction receives a zero dual value in one solution, given the operation of a set of other constraints, and a positive dual price is imputed to it in another solution, in which it plays a direct role. Our Problems 1a and 1c are cases in point. The restriction $x_2 \geq 0$ received a zero dual in the former of these two, while this value rises to $\frac{1}{4}$ in the latter. In contrast, the dual associated with the upper bound $x_1 \leq 1$ receives a positive value in Problem 1a, while this (automatically) drops to zero when the constraint is omitted in Problem 1c. This situation may be labeled one of "relative redundancy."

There are special cases which Gomory and Baumol present in which the problem of redundancy is abated somewhat. This subject is the concern of our next section.

5.5. Uniqueness of the Recomputed Dual Variables

Under certain special circumstances the recomputed duals are independent of the particular choices of cutting planes employed to gain the integer optimum. These situations are described by Gomory and Baumol as follows:

> Although the recomputed prices depend on the actual course of the calculation, . . . there is one case in which a type of uniqueness prevails.
>
> If there is some set of n original inequalities such that these n alone determine the same integer solution as does the full set of inequalities, then, if all other inequalities are dropped, the recomputed prices for the reduced problem are unique, and are identical with the prices obtained by solving the reduced problem *as an ordinary noninteger linear programming problem.*[23]
>
> To obtain the prices in question we simply ignore the constraints other than the n singled out, and proceed to solve the problem first as an ordinary noninteger linear programming problem, and then, as usual, by adding new inequalities to obtain the integer solution. At the solution point of the ordinary linear programming problem, all n inequalities are satisfied as equalities, i.e., all the slack variables are nonbasic (we may regard x as being the slack of the inequality $-x \leq 0$, for if we

[23]Gomory and Baumol, *op. cit.*, p. 533. (Italics supplied.) n here is the number of original nonbasic variables.

> introduce an x', with $-x + x' = 0$, we have $x = x'$). This point may well not satisfy the inequalities that we are temporarily disregarding. This fact makes no difference, however, when it comes to generating the new inequalities whose validity is not affected. Of course when we reach the final integer solution, which is the same for the problem with or without these inequalities, the disregarded inequalities will be satisfied by hypothesis.[24]

The situation encompassed by the quotations above may be illustrated by a series of numerical examples. One of these has already been encountered as a version of our first problem, namely Problem 1c (Table 5.6). As mentioned in another context, the two unity upper bound constraints of that problem are unnecessary for achieving the integer optimum because the budget constraint alone is sufficiently restrictive to rule out multiples of the projects. When the problem is solved in this form, the integer solution is the same as the one which was obtained in the complete formulation, Problem 1a, in which the upper bounds were stated explicitly.

In the reduced version of the problem the number of constraints is apparently less than the number of original nonbasic variables: the single budget constraint, and the variables x_1 and x_2. A closer examination of the calculations reveals, however, that another constraint is being applied, albeit implicitly. When x_1 enters the basis in Tableau 2, its level is determined by the reduction of x_2 to zero. Although increasing x_1 above $\frac{5}{4}$, its value in Tableau 2, would be "profitable," to do so would make x_2 negative. But x_2 becomes nonbasic in this tableau, hence its value must be zero. However, from this point on, x_2 is presumed to be nonnegative for an additional reason.

In order that the cutting plane

$$s_1 = -\frac{3}{4} - \frac{3}{4}(-\bar{x}_1) - \frac{1}{4}(-x_2)$$

actually be a restriction, we must require that both $\bar{x}_1 \geq 0$ and $x_2 \geq 0$. Without these conditions the cutting plane would not be a constraint which an integer solution to the original problem must satisfy. Hence, from the linear programming optimum on, the variables which are nonbasic at this iteration are required to be nonnegative. This implies that the restrictions represented by nonbasic slack variables must be satisfied,

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[24]Gomory and Baumol, *op. cit.*, pp. 548-49. The original gave the quantity n as N.

an ordinary (that is, original nonbasic) variable being interpreted as the slack variable of its nonnegativity constraint.[25]

The recomputation at the end of Table 5.6 shows the resulting dual prices to be $\pi_1^* = \frac{3}{4}$, $\pi_3^* = \frac{1}{4}$; that is, $\rho^* = \frac{3}{4}$, $\gamma_2^* = \frac{1}{4}$. These values also appear in the zero row of Tableau 2, the linear programming optimum of Problem 1c, in the columns headed $\bar{x}_1$ and x_2 respectively. Thus in this example positive dual values are imputed to exactly *two* constraints, the budget constraint and the nonnegativity constraint on x_2, both of which are utilized in determining the requirements space of the linear programming optimum. Both of these restrictions were required to generate the cutting plane which yielded the integer optimum.

Another facet of this special case yielding a "kind of uniqueness" for the recomputed duals is illustrated by the following example.

Problem 3

$$\begin{aligned} &\text{Maximize} && 3x_1 + 2x_2 \\ &\text{Subject to (a)} && 10x_1 + 5x_2 \leq 11 \\ &\text{(b)} && x_1 \leq 1 \\ &\text{(c)} && x_2 \leq 1 \\ &\text{(d)} && x_1, x_2 \geq 0. \end{aligned} \tag{5.31}$$

In this problem upper bound (5.31b) is clearly not needed for obtaining the integer solution. This restriction has been omitted in Table 5.7, in which this example has been solved by use of the Method of Integer Forms as Problem 3a. The recomputation is carried out at the end of the table.

The imputed duals arrived at by recomputation are $\pi_1^* = \frac{3}{10}$, $\pi_2^* = \frac{3}{2}$, $\pi_4^* = 1$ ($\rho^* = \frac{3}{10}$, $\mu_2^* = \frac{3}{2}$, $\gamma_2^* = 1$). Evidently, positive dual values are imputed to three constraints, although only two constraints receive such values in the linear programming optimal solution, and these are different: $\pi_1^* = \frac{3}{10}$, $\pi_2^* = \frac{1}{2}$ ($\rho^* = \frac{3}{10}$, $\mu_2^* = \frac{1}{2}$). This discrepancy may be accounted for by closer examination of the calculations of the Method of Integer Forms, as applied to this problem.

One normally proceeds by pivoting until the problem is both primal and dual feasible, hence optimal. Once this point is reached one adds the cutting plane constraint and reoptimizes. This process involves removing the initial infeasibility brought about at this point by the negativity of

---

[25]Figure 5.1 shows that the constraint $x_2 \geq 0$ is operative in this example.

TABLE 5.7. PROBLEM 3a AND SOLUTION

Maximize $3x_1 + 2x_2$
Subject to $10x_1 + 5x_2 \leq 11$
$x_2 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−3	−2
$\bar{x}_1$ =	11	10*	5
$\bar{x}_2$ =	1		1

Tableau 2

	1	$-\bar{x}_1$	$-x_2$
z =	33/10	3/10	−1/2
x_1 =	11/10	1/10	1/2
$\bar{x}_2$ =	1		1*

Tableau 3

	1	$-\bar{x}_1$	$-\bar{x}_2$
z =	19/5	3/10	1/2
→ x_1 =	3/5	1/10	−1/2
x_2 =	1		1
s_1 =	−3/5	−1/10	−1/2*

(L.P. optimum)

Tableau 4

	1	$-\bar{x}_1$	$-s_1$
z =	16/5	1/5	1
x_1 =	6/5	1/5	−1
x_2 =	−1/5	−1/5*	2
$\bar{x}_2$ =	6/5	1/5	−2

Tableau 5

	1	$-x_2$	$-s_1$
z =	3	1	3
x_1 =	1	1	1
$\bar{x}_1$ =	1	−5	−10
$\bar{x}_2$ =	1	1	

(Integer optimum)

Recomputation

	π	π^*
$\bar{x}_1$	0	$0 + (1/10)3 = 3/10$
$\bar{x}_2$	0	$0 + (1/2)3 = 3/2$
x_1	0	$0 = 0$
x_2	1	$1 = 1$
s_1	3	—

Cutting Plane s_1:

$$s_1 = -\frac{3}{5} + \frac{1}{10}\bar{x}_1 + \frac{1}{2}\bar{x}_2 \geq 0$$

$$= -\frac{3}{5} + \frac{1}{10}(11 - 10x_1 - 5x_2) + \frac{1}{2}(1 - x_2) \geq 0$$

that is, $$x_1 + x_2 \leq 1$$

the slack of the cutting plane. In removing this infeasibility one may not return to feasibility in a single step. This is the case in Tableau 4 of Table 5.7, in which $x_2 = -\frac{1}{5}$. In the usual procedure this infeasibility is removed before the addition of another cutting plane. The result of this pivoting is displayed in Tableau 5 which is the integer optimum.

One fact is overlooked in carrying out this procedure. In removing the infeasibility on x_2, a nonnegativity restriction on this variable is being imposed. This restriction was not among those utilized in the

linear programming optimum. Hence the solution of Table 5.7 made use of three constraints, or one more than the number of original nonbasic constraints, and hence the conditions for "uniqueness" have been violated.

Imposition of the constraint $x_2 \geq 0$ is not necessary, however, for the attainment of the integer optimum, as reference to Fig. 5.3 will show. In this figure Problem 3a has been plotted without the unneeded constraint $x_1 \leq 1$. The initial linear programming solution is at point A, with coordinates $x_1 = \frac{3}{5}$, $x_2 = 1$. The first cutting plane, expressed in terms of the variables plotted (as derived in the table), is $x_1 + x_2 \leq 1$. Without imposition of the restriction $x_2 \geq 0$, the new optimum is at B, with coordinates $x_1 = \frac{6}{5}$, $x_2 = -\frac{1}{5}$. When the nonnegativity constraint is then imposed, that is, the infeasibility is removed, the linear programming optimum including these constraints shifts to point D which is the integer optimum: $x_1^* = 1$, $x_2^* = 0$.

An alternative to the use of the constraint $x_2 \geq 0$ is the imposition of

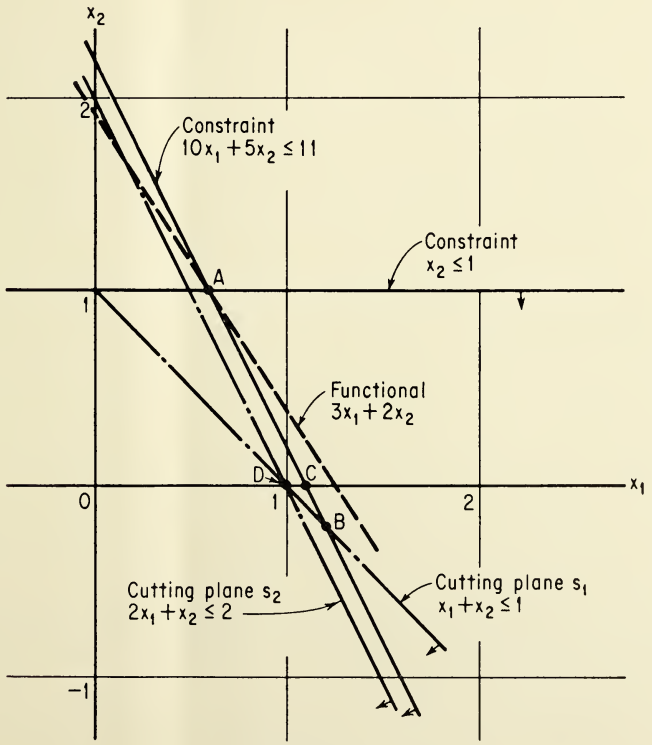

Figure 5.3.

TABLE 5.8. PROBLEM 3b AND SOLUTION

Maximize $3x_1 + 2x_2$
Subject to $10x_1 + 5x_2 \leq 11$
$x_2 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−3	−2
$\bar{x}_1$ =	11	10*	5
$\bar{x}_2$ =	1		1

Tableau 2

	1	$-\bar{x}_1$	$-x_2$
z =	33/10	3/10	−1/2
x_1 =	11/10	1/10	1/2
$\bar{x}_2$ =	1		1*

Tableau 3

	1	$-\bar{x}_1$	$-\bar{x}_2$
z =	19/5	3/10	1/2
→ x_1 =	3/5	1/10	−1/2
x_2 =	1		1
s_1 =	−3/5	−1/10	−1/2*

(L.P. optimum)

Tableau 4

	1	$-\bar{x}_1$	$-s_1$
z =	16/5	1/5	1
→ x_1 =	6/5	1/5	−1
x_2 =	−1/5	−1/5	2
$\bar{x}_2$ =	6/5	1/5	−2
s_2 =	−1/5	−1/5*	

Tableau 5

	1	$-s_2$	$-s_1$
z =	3	1	1
x_1 =	1	1	−1
x_2 =		−1	2
$\bar{x}_2$ =	1	1	−2
$\bar{x}_1$ =	1	−5	

(Integer optimum)

Recomputation

	π	π'	π^*
$\bar{x}_1$	0	0 + (1/5)1	1/5 + (1/10)1 = 3/10
$\bar{x}_2$	0	0	0 + (1/2)1 = 1/2
x_1	0	0	0 = 0
x_2	0	0	0 = 0
s_1	1	1	—
s_2	1	—	—

Cutting plane s_2:

$$s_2 = -\frac{1}{5} + \frac{1}{5}\bar{x}_1 \geq 0$$

$$= -\frac{1}{5} + \frac{1}{5}(11 - 10x_1 - 5x_2) \geq 0$$

that is, $$2x_1 + x_2 \leq 2.$$

a further cutting plane at Tableau 4 of Table 5.7. This has been done in Table 5.8, which repeats Table 5.7 up to this stage, but adds the restriction

$$s_2 = -\frac{1}{5} + \frac{1}{5}\bar{x}_1 \geq 0$$

in Tableau 4. This cutting plane, which is equivalent to the restriction $2x_1 + x_2 \leq 2$, forces the solution to the integer optimum, as both Tableau 5 in Table 5.8 and Fig. 5.3 reveal. The nonnegativity constraint is thus seen not to be necessary, that is, the two explicitly stated constraints, plus the cutting planes which these two alone generate, are sufficient to yield the integer optimal solution. As the recomputation at the end of Table 5.8 shows, the duals which the alternate calculations carried out in Problem 3b yield are, indeed, the duals of the linear programming optimum of Tableau 3.

To show why this is the case and in what sense these recomputed dual prices are unique may be explained in the following way. The linear programming optimum solution is determined by the operation of exactly n restrictions (in the absence of degeneracy), the number n being the dimensionality of the requirements space. Among these restrictions may be some nonnegativity constraints. If these are sufficient to yield the integer optimum, then this means that no other restrictions are needed to generate all the cutting planes required for solving the integer problem. Specifically, every pivot element subsequent to the original linear programming optimum is an element in one of the cutting planes. Thus, at each subsequent iteration a cutting plane is added and its slack is made nonbasic. Primal infeasibility generated in this process, except in the slack of the cutting planes, is ignored. If this process leads to the integer optimum, then the process of recomputation consists merely of "backward pivoting," from the integer optimum to the original linear programming optimum, in which the path of cutting planes utilized to obtain the integer optimum is retraced exactly.

The backward pivoting process reveals all the properties claimed for these "unique" duals. First, since the end of the process is the original linear programming optimum, only the constraints binding at the linear programming optimum receive possibly nonzero dual values. In fact, since the coefficients of the cutting planes (in terms of the negatives of the currently nonbasic variables, as in the condensed format used in the tables) are all nonpositive, the duals of all slack variables which have remained nonbasic from the linear programming to the integer programming optimum are all increased, iteration by iteration, when going backward from the final to the linear programming optimum tableau. Similarly, all slack variables (including the slack variables of nonnegativity constraints) which become nonbasic in the backward path will have their dual values increased above their zero level (except in the instance of dual degeneracy). Further, since it does not matter which set of cutting planes is chosen for obtaining the integer optimum, the back-

TABLE 5.9. PROBLEM 4a AND SOLUTION

Maximize $3x_1 + 8x_2$
Subject to $2x_1 + 6x_2 \leq 9$
$x_1 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−3	−8
$\bar{x}_1$ =	9	2	6
$\bar{x}_2$ =	1	1*	

Tableau 2

	1	$-\bar{x}_2$	$-x_2$
z =	3	3	−8
$\bar{x}_1$ =	7	−2	6*
x_1 =	1	1	

Tableau 3

	1	$-\bar{x}_2$	$-\bar{x}_1$
z =	37/3	1/3	4/3
→ x_2 =	7/6	−1/3	1/6
x_1 =	1	1	
s_1 =	−1/6	−2/3*	−1/6

(L.P. optimum)

Tableau 4

	1	$-s_1$	$-\bar{x}_1$
z =	49/4	1/2	5/4
x_2 =	5/4	−1/2	1/4
→ x_1 =	3/4	3/2	−1/4
$\bar{x}_2$ =	1/4	$-3/2^0$	1/4
s_2 =	−3/4	−1/2*	−3/4

Tableau 5

	1	$-s_2$	$-\bar{x}_1$
z =	23/2	1	1/2
x_2 =	2	−1	1
→ x_1 =	−3/2	3	−5/2
$\bar{x}_2$ =	5/2	−3	5/2
s_1 =	3/2	-2^0	3/2
s_3 =	−1/2		−1/2*

Tableau 6

	1	$-s_2$	$-s_3$
z =	11	1	1
x_2 =	1	−1	2
x_1 =	1	3	−5
$\bar{x}_2$ =		−3	5
s_1 =		−2	3
$\bar{x}_1$ =	1		-2^0

(Integer optimum)

Recomputation

	π	π'	π''	π^*
$\bar{x}_1$	0	0 + (1/2)1	(1/2) + (3/4)1	(5/4) + (1/6)(1/2) = 4/3
$\bar{x}_2$	0	0	0	0 + (2/3)(1/2) = 1/3
x_1	0	0	0	0 = 0
x_2	0	0	0	0 = 0
s_1	0	0	0 + (1/2)1	—
s_2	1	1	—	—
s_3	1	—	—	—

ward pivoting process always ends with the linear programming optimal dual values. In that sense the duals so arrived at are unique — independent of the path followed in obtaining the integer solution.

To illustrate these remarks we utilize another example which is solved by imposition of different sets of cutting planes. The problem is

Problem 4

$$\begin{aligned} &\text{Maximize} && 3x_1 + 8x_2 \\ &\text{Subject to (a)} && 2x_1 + 6x_2 \leq 9 \\ &\text{(b)} && x_1 \leq 1 \\ &\text{(c)} && x_2 \leq 1 \\ &\text{(d)} && x_1, x_2 \geq 0. \end{aligned} \tag{5.32}$$

The budget constraint of this problem makes restriction (5.32c) superfluous, and this has been omitted from the calculations in Problems 4a and 4b, Tables 5.9 and 5.10, respectively. Similarly, as both sets of

TABLE 5.10. PROBLEM 4b AND SOLUTION

Maximize $3x_1 + 8x_2$
Subject to $2x_1 + 6x_2 \leq 9$
$x_1 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
$z =$		-3	-8
$\bar{x}_1 =$	9	2	6
$\bar{x}_2 =$	1	1*	

Tableau 2

	1	$-\bar{x}_2$	$-x_2$
$z =$	3	3	-8
$\bar{x}_1 =$	7	-2	6*
$x_1 =$	1	1	

Tableau 3

	1	$-\bar{x}_2$	$\bar{x}_1$
$\rightarrow z =$	37/3	1/3	4/3
$x_2 =$	7/6	$-1/3$	1/6
$x_1 =$	1	1	
$s_1 =$	$-1/3$	$-1/3$*	$-1/3$

(L.P. optimum)

Tableau 4

	1	$-s_1$	$-\bar{x}_1$
$z =$	12	1	1
$\rightarrow x_2 =$	3/2	-1	1/2
$x_1 =$		3	-1
$\bar{x}_2 =$	1	-3^0	1
s_2	$-1/2$		$-1/2$*

Tableau 5

	1	$-s_1$	$-s_2$
$z =$	11	1	2
$x_2 =$	1	-1	1
$x_1 =$	1	3	-2
$\bar{x}_2 =$		-3	2
$\bar{x}_1 =$	1		-2^0

(Integer optimum)

Recomputation

	π	π'	π^*
$\bar{x}_1$	0	$0 + (1/2)2$	$1 + (1/3)1 = 4/3$
$\bar{x}_2$	0	0	$0 + (1/3)1 = 1/3$
x_1	0	0	$0 = 0$
x_2	0	0	$0 = 0$
s_1	1	1	—
s_2	2	—	—

computations bear out, none of the nonnegativity constraints is used either explicitly or implicitly. The resulting recomputations in each case yield the same dual values as the duals of the linear programming optimum.

These examples also serve to illustrate the similarity between the recomputation process and "backward pivoting." In Tableau 6 of Table 5.9, the integer optimum, if $\bar{x}_1$ is removed from the basis and s_3 is introduced, the resulting tableau is precisely Tableau 5. The dual value now associated with $\bar{x}_1$ is now $1/[-1(-2)] = \frac{1}{2}$, which is the result of dividing the zero row element in the pivot column of Tableau 6 by the negative of the pivot element (indicated by the circle). It is also the result of the first step in the recomputation, which the table at the end of these calculations reveals. The element in the zero row under s_2 in Tableau 5 is the same as it is in Tableau 6 because of the presence of a zero in the pivot row and s_2 column of Tableau 6.

At the next backward iteration, s_1 is removed from the basis and s_2 enters. The dual value of $\bar{x}_1$ then becomes $\frac{1}{2} - 1\frac{(\frac{3}{2})}{(-2)} = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$. This again repeats these calculations in the recomputation, as found in the third column. The other duals at this and the following backward iteration are obtained in similar fashion and also recapitulate the recomputation process. Examination of Table 5.10 shows that, although the cutting planes employed to obtain the integer optimum differ here from the previous table (as do the duals before imputation), the process of backward pivoting reconstructs the recomputation calculations, ending with the duals of the linear programming optimum.

5.6. Limitations of the Uniqueness Property

As we have just observed, in the special circumstances in which n constraints alone suffice to generate all the cutting planes needed for the integer optimum, a process of backward pivoting accomplishes the recomputation process, and the duals so obtained are exactly the duals obtaining in the linear programming optimum. "Now the expression for the variables in terms of the nonbasic set [of variables at the linear programming optimum] is unique, so all the coefficients must be identical with those obtained in solving the n inequalities of an ordinary linear programming problem so that the ordinary linear programming price and the redistributed price coincide for input goods."[26]

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[26]Gomory and Baumol, *op. cit.*, p. 550.

Some of the problems we have examined in which this special uniqueness applies also reveal a weakness of these "unique" recomputed duals which is difficult to overcome by the present approach. Problem 4 reveals one facet of the difficulty. If this problem is solved in a straightforward manner, without regard to the number of restrictions (that is, without elimination of any of the upper bound constraints), the linear program-

TABLE 5.11. PROBLEM 4c AND SOLUTION

Maximize $3x_1 + 8x_2$
Subject to $2x_1 + 6x_2 \leq 9$
$x_1 \leq 1$
$x_2 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−3	−8
$\bar{x}_1$ =	9	2	6
$\bar{x}_2$ =	1	1*	
$\bar{x}_3$ =	1		1

Tableau 2

	1	$-\bar{x}_2$	$-x_2$
z =	3	3	−8
$\bar{x}_1$ =	7	−2	6
x_1 =	1	1	
$\bar{x}_3$ =	1		1*

Tableau 3

	1	$-\bar{x}_2$	$-\bar{x}_3$
z =	11	3	8
$\bar{x}_1$ =	1	−2	−6
x_1 =	1	1	
x_2 =	1		1

(L. P. optimum and Integer optimum)

ming solution is identically the integer solution, as shown in Table 5.11. Recomputation is not required since no cutting planes were utilized in obtaining the integer solution. The integer and ordinary linear programming dual prices coincide. However, as comparison with either Table 5.9 or 5.10 reveals, the numerical values differ from those obtained earlier.[27]

There is a more serious situation, due to its ubiquity, in which the possibility of backward pivoting is not sufficient to yield truly unique duals. It comes about when more than one subset of n constraints

[27]The absence of degeneracy in this linear programming solution indicates that these dual prices are unique.

suffices to yield the integer optimum. We have encountered an instance of this in our first example. In Problem 1c, the budget constraint and nonnegativity on x_2 were shown to suffice for obtaining the integer optimum. Our discussion of the geometry of this example and Fig. 5.1 also brought out that an alternative to the restriction $x_2 \geq 0$ is the

TABLE 5.12. PROBLEM 1d AND SOLUTION

Maximize $3x_1 + 2x_2$
Subject to $4x_1 + 3x_2 \leq 5$
$x_1 \leq 1$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−3	−2
$\bar{x}_1$ =	5	4	3
$\bar{x}_2$ =	1	1*	

Tableau 2

	1	$-\bar{x}_2$	$-x_2$
z =	3	3	−2
$\bar{x}_1$ =	1	−4	3*
x_1 =	1	1	

Tableau 3

	1	$-\bar{x}_2$	$-\bar{x}_1$
z =	11/3	1/3	2/3
→ x_2 =	1/3	−4/3	1/3
x_1 =	1	1	
s_1 =	−1/3	−2/3*	−1/3

(L. P. optimum)

Tableau 4

	1	$-s_1$	$-\bar{x}_1$
z =	7/2	1/2	1/2
x_2 =	1	−2	1
→ x_1 =	1/2	3/2	−1/2
$\bar{x}_2$ =	1/2	−3/2	1/2
s_2 =	−1/2	−1/2	−1/2*

Tableau 5

	1	$-s_1$	$-s_2$
z =	3		1
x_2 =		−3	2
x_1 =	1	2	−1
$\bar{x}_2$ =		−2	1
$\bar{x}_1$ =	1	1	−2

(Integer optimum)

Recomputation

	π	π'	π^*
$\bar{x}_1$	0	0 + (1/2)1	1/2 + (1/3)(1/2) = 2/3
$\bar{x}_2$	0	0	0 + (2/3)(1/2) = 1/3
x_1	0	0	0 = 0
x_2	0	0	0 = 0
s_1	0	0 + (1/2)1	—
s_2	1	—	—

constraint $x_1 \leq 1$. (To facilitate comparison with the earlier example, we have repeated the calculations of Table 5.1, omitting the unnecessary constraint $x_2 \leq 1$, in Table 5.12 as Problem 1d). The dual values obtained in the two versions, Problems 1c and 1d, are different, necessarily imputing positive prices, at least in part, to different constraints.

The same situation is revealed more strikingly when Problems 1 and 3 are combined into a single, two-period Lorie-Savage type problem, as follows.

Problem 5

$$\begin{aligned} &\text{Maximize} && 3x_1 + 2x_2 \\ &\text{Subject to (a)} && 4x_1 + 3x_2 \leq 5 \\ &\text{(b)} && 10x_1 + 5x_2 \leq 11 \\ &\text{(c)} && x_1 \leq 1 \\ &\text{(d)} && x_2 \leq 1 \\ &\text{(e)} && x_1 \geq 0 \\ &\text{(f)} && x_2 \geq 0. \end{aligned} \tag{5.33}$$

A solution to this complete problem is presented in Table 5.13, as Problem 5a.[28] The recomputation carried out at the end of the table yields the values $\rho_1^* = \frac{3}{10}$, $\rho_2^* = \frac{9}{50}$, $\mu_1^* = 0$, and $\mu_2^* = \frac{1}{5}$. In Tableau 6 pivoting is done in the $\bar{x}_4$ row, to remove the infeasibility indicated by the negative value of this variable. An alternative procedure, not strictly conforming to the Method of Integer Forms (though a valid variant of it used in some computer codes)[29] is to add a cutting plane at this iteration, and to pivot in the new row. This has been done in Table 5.14. Although not lexicographically dual feasible, the optimal integer solution is obtained at the very next tableau. More important, however, is that by utilizing a cutting plane rather than removing the infeasibility in an original slack variable, the process conforms to the requirement for "backward pivoting" so that the recomputed duals (bottom of Table 5.14) are identically the duals of the linear programming solution of Tableau 4 of Problem 5a.

Several conclusions may be drawn from this example. First, although the calculations were begun with more than n constraints, necessarily only n were binding at the linear programming optimum.[30] The constraints which were binding at this point were sufficient to generate all

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[28]A further cutting plane is required at Tableau 8 to yield an all integer matrix — which is not needed for our purposes.

[29]For example, IP3 (or IPM3), a code written by R. Gomory and R. Levitan at IBM.

[30]Except for degeneracy, only n slack variables (including slacks of nonnegativity constraints) receive positive dual values at the linear programming optimum. This is the number of binding constraints.

TABLE 5.13. PROBLEM 5a AND SOLUTION

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 2x_2 \\ \text{Subject to} \quad & 4x_1 + 3x_2 \leq 5 \\ & 10x_1 + 5x_2 \leq 11 \\ & x_1 \leq 1, \quad x_2 \leq 1 \end{aligned}$$

Tableau 1

	1	$-x_1$	$-x_2$
z =		-3	-2
$\bar{x}_1$ =	5	4	3
$\bar{x}_2$ =	11	10	5
$\bar{x}_3$ =	1	1*	
$\bar{x}_4$ =	1		1

Tableau 2

	1	$-\bar{x}_3$	$-x_2$
z =	3	3	-2
$\bar{x}_1$ =	1	-4	3
$\bar{x}_2$ =	1	-10	5*
x_1 =	1	1	
$\bar{x}_4$ =	1		1

Tableau 3

	1	$-\bar{x}_3$	$-\bar{x}_2$
z =	17/5	-1	2/5
$\bar{x}_1$ =	2/5	2*	$-3/5$
x_2 =	1/5	-2	1/5
x_1 =	1	1	
$\bar{x}_4$ =	4/5	2	$-1/5$

Tableau 4

	1	$-\bar{x}_1$	$-\bar{x}_2$
z =	18/5	1/2	1/10
$\bar{x}_3$ =	1/5	1/2	$-3/10$
→ x_2 =	3/5	1	$-2/5$
x_1 =	4/5	$-1/2$	3/10
$\bar{x}_4$ =	2/5	-1	2/5
s_1 =	$-3/5$		$-3/5$*

(L. P. optimum)

Tableau 5

	1	$-\bar{x}_1$	$-s_1$
z =	7/2	1/2	1/6
→ $\bar{x}_3$ =	1/2	1/2	$-1/2$
x_2 =	1	1	$-2/3$
x_1 =	1/2	$-1/2$	1/2
$\bar{x}_4$ =		-1	2/3
$\bar{x}_2$ =	1		$-5/3$
s_2 =	$-1/2$	$-1/2$	$-1/2$*

Tableau 6

	1	$-\bar{x}_1$	$-s_2$
z =	10/3	1/3	1/3
$\bar{x}_3$ =	1	1	-1
x_2 =	5/3	5/3	$-4/3$
x_1 =		-1	1
$\bar{x}_4$ =	$-2/3$	$-5/3$*	4/3
$\bar{x}_2$ =	8/3	5/3	$-10/3$
[s_1 =	1	1	-2]

TABLE 5.13 (cont.). PROBLEM 5a AND SOLUTION

Tableau 7

	1	$-\bar{x}_4$	$-s_2$
z =	16/5	1/5	3/5
→ $\bar{x}_3$ =	3/5	3/5	−1/5
x_2 =	1	1	
x_1 =	2/5	−3/5	1/5
$\bar{x}_1$ =	2/5	−3/5	−4/5
$\bar{x}_2$ =	2	1	−2
s_3 =	−3/5	−3/5*	−4/5

Tableau 8

	1	$-s_3$	$-s_2$
z =	3	1/3	1/3
$\bar{x}_3$ =		1	−1
x_2 =		5/3	−4/3
x_1 =	1	−1	1
$\bar{x}_1$ =	1	−1	
$\bar{x}_2$ =	1	5/3	−10/3
$\bar{x}_4$ =	1	−5/3	4/3

(Integer optimum)

Recomputation

	π	π'	π''	π^*
$\bar{x}_1$	0	0	0 + (1/2)(3/5)	3/10 = 3/10
$\bar{x}_2$	0	0	0	0 + (3/5)(3/10) = 9/50
$\bar{x}_3$	0	0	0	0 = 0
$\bar{x}_4$	0	0 + (3/5)(1/3)	1/5	1/5 = 1/5
x_1	0	0	0	0 = 0
x_2	0	0	0	0 = 0
s_1	0	0	0 + (1/2)(3/5)	—
s_2	1/3	1/3 + (4/5)(1/3)	—	—
s_3	1/3	—	—	—

the cutting planes needed to arrive at the integer solution. In Problem 5 these were the two budget constraints. Had the upper bound constraints (and nonnegativity constraints) been omitted from the initial tableau, the linear programming and integer programming solutions would have been the same as they are in Tables 5.13 and 5.14.[31]

In addition to the "unique" duals obtained in Table 5.14, we have already seen that constraints (5.33a and c) alone yield the integer solution to this problem (Problem 1d) and so do constraints (5.33a and f) (Problem 1c). Constraints (5.33b and d) (Problem 3b) are also sufficient, as was shown in Table 5.8. The four sets of "unique" dual prices and two "nonunique" sets are summarized in Table 5.15, which could, no doubt, be expanded.

The situation in the example above in which more than one subset of n constraints was sufficient to yield the integer optimum is analogous to degeneracy in ordinary linear programming. The difference between

[31]With variables $\bar{x}_3$ and $\bar{x}_4$ omitted, of course.

TABLE 5.14. PROBLEM 5b AND SOLUTION

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 2x_2 \\ \text{Subject to} \quad & 4x_1 + 3x_2 \leq 5 \\ & 10x_1 + 5x_2 \leq 11 \\ & x_1 \leq 1, \quad x_2 \leq 1 \end{aligned}$$

Tableau 1

	1	$-x_1$	$-x_2$
z =		−3	−2
$\bar{x}_1$ =	5	4	3
$\bar{x}_2$ =	11	10	5
$\bar{x}_3$ =	1	1*	
$\bar{x}_4$ =	1		1

Tableau 2

	1	$-\bar{x}_3$	$-x_2$
z =	3	3	−2
$\bar{x}_1$ =	1	−4	3
$\bar{x}_2$ =	1	−10	5*
x_1 =	1	1	
$\bar{x}_4$ =	1		1

Tableau 3

	1	$-\bar{x}_3$	$-\bar{x}_2$
z =	17/5	−1	2/5
$\bar{x}_1$ =	2/5	2*	−3/5
x_2 =	1/5	−2	1/5
x_1 =	1	1	
$\bar{x}_4$ =	4/5	2	−1/5

Tableau 4

	1	$-\bar{x}_1$	$-\bar{x}_2$
z =	18/5	1/2	1/10
$\bar{x}_3$ =	1/5	1/2	−3/10
→ x_2 =	3/5	1	−2/5
x_1 =	4/5	−1/2	3/10
$\bar{x}_4$ =	2/5	−1	2/5
s_1 =	−3/5		−3/5*

(L. P. optimum)

Tableau 5

	1	$-\bar{x}_1$	$-s_1$
z =	7/2	1/2	1/6
→ $\bar{x}_3$ =	1/2	1/2	−1/2
x_2 =	1	1	−2/3
x_1 =	1/2	−1/2	1/2
$\bar{x}_4$ =		−1	2/3
$\bar{x}_2$ =	1		−5/3
s_2 =	−1/2	−1/2	−1/2*

Tableau 6

	1	$-\bar{x}_1$	$-s_2$
→ z =	10/3	1/3	1/3
$\bar{x}_3$ =	1	1	−1
x_2 =	5/3	5/3	−4/3
x_1 =		−1	1
$\bar{x}_4$ =	−2/3	−5/3	4/3
$\bar{x}_2$ =	8/3	5/3	−10/3
s_1 =	1	1	−2
s_3 =	−1/3	−1/3	−1/3

TABLE 5.14 (cont.). PROBLEM 5b AND SOLUTION

Tableau 7

		1	$-s_2$	$-s_2$
z	=	3	1	
$\bar{x}_3$	=		3	−2
x_2	=		5	−3
x_1	=	1	−3	2
$\bar{x}_4$	=	1	−5	3
$\bar{x}_2$	=	1	5	−5
$\bar{x}_1$	=	1	−3	1

(Integer optimum)

Recomputation

	π	π'	π''	π^*
$\bar{x}_1$	0	0 + (1/3)1	1/3 + (1/2)(1/3)	1/2 = 1/2
$\bar{x}_2$	0	0	0	0 + (3/5)(1/6) = 1/10
$\bar{x}_3$	0	0	0	0 = 0
$\bar{x}_4$	0	0	0	0 = 0
x_1	0	0	0	0 = 0
x_2	0	0	0	0 = 0
s_1	0	0	0 + (1/2)(1/3)	—
s_2	0	0 + (1/3)1	—	—
s_3	1	—	—	—

them arises in the following way. Degeneracy in linear programming occurs when several subsets of restrictions yield the optimum separately. In integer programming it is only necessary that several subsets of constraints suffice to generate the cutting planes needed to obtain the integer optimum. These subsets may yield different linear programming optima and may admit different sets of lattice points. However, the integer optimum point is common to the feasible region included by all of them and will be optimal for all of them.

The preceding several sections of this chapter have been employed to examine and enlarge upon the nature of the process which Gomory and Baumol propose as a means of obtaining dual evaluators of integer programming constraints. These quantities, which they have called the recomputed duals, assign values exclusively to the original constraints, distributing to these the duals associated with the additional constraints of the cutting planes. We may summarize the discussion so far by listing the properties of the recomputed duals. First, their numerical values generally depend on the choice of cutting planes, that is, on the path

TABLE 5.15. SUMMARY OF RECOMPUTATIONS FOR PROBLEM 5

Problem No.	1c[a]	1d[a]	3b[a]	3a	5b[a]	5a
Table No.	5.6	5.12	5.8	5.7	5.14	5.13
ρ_1	3/4	2/3	0	0	1/2	3/10
ρ_2	0	0	3/10	3/10	1/10	9/50
μ_1	0	1/3	0	0	0	0
μ_2	0	0	1/2	3/2	0	1/5
γ_1	0	0	0	0	0	0
γ_2	1/4	0	0	1/2	0	0

[a]These duals are independent of the cutting planes used in the particular model, hence they satisfy the conditions for "uniqueness."

utilized in the computations, and an explicit record of the added constraints is needed for performing the recomputation. Second, they share with the duals of linear programming the property that they "price out" the utilized activities so as to leave them profitless, while assigning losses to activities rejected in the optimal solution. In doing so they may assign "subsidies" to some activities. Third, they do not measure the effects of discrete changes in the stipulations vector[32] and, in general, do not satisfy the usual duality relation — value of the primal solution equals value of the dual solution. Hence they function only partially as shadow prices. Fourth, they do assign a zero value to constraints whose complete omission would not alter the integer optimum. A zero value may be assigned to a constraint which is "absolutely redundant," that is, one whose omission from any subset of original constraints would not alter the ability of the subset to produce the integer optimum by use of the MIF algorithm. Alternatively, a zero value may be assigned to a constraint which is redundant only given the other constraints, and hence it may receive a positive dual value in a solution utilizing a different subset of constraints. Fifth, whenever a subset of n original constraints is sufficient to generate the required cutting planes, and hence yields the same integer optimum as would the complete set of restrictions, then the recomputation corresponds to the process of backward pivoting, and the duals so obtained are independent of the choice of cutting planes. However, even in this situation it may be that several subsets of n constraints are sufficient, in which case more than one set of "unique" duals results.

We shall shortly propose an alternative method for generating values which may serve as dual evaluators of the original constraints. In the meantime we briefly examine the recomputation, as proposed by Gomory

[32]That is, in terms of the right side.

and Baumol, of the duals to the Lorie-Savage problem.

5.7. Recomputed Dual Prices and the Lorie-Savage Model

The integer programming model of the Lorie-Savage problem given by equations (4.1) (in which there are only independent projects and nonnegative C_t, c_{tj}) always yields an integer solution. The upper bound constraints on the projects confine the problem to the unit cube, which is further cut down by the budget hyperplanes whose coefficients are nonnegative. Despite the simplicity of this structure, it is not possible to assert that at least one set of recomputed dual prices may be obtained by solving an ordinary linear programming problem.

It would indeed be tempting to argue that the original number of constraints of the model, $T + n$ (where T is the number of time periods for which budgets are stated, and n is the number of projects), can be reduced to n automatically. Specifically, it would seem reasonable that the upper bound constraints on those projects which the integer solution rejects could be omitted, assuming only that these latter can be identified.[33] Such is not the case, however, as the following example will demonstrate. Consider Problem 6:

Problem 6

$$\begin{aligned} &\text{Maximize} && 5x_1 + 4x_2 + 3x_3 \\ &\text{Subject to} && \text{(a)} \quad 6x_1 + 4x_2 + 4x_3 \leq 13 \\ &&& \text{(b)} \quad x_1 \leq 1 \\ &&& \text{(c)} \quad x_2 \leq 1 \\ &&& \text{(d)} \quad x_3 \leq 1 \\ &&& \text{(e)} \quad x_1, x_2, x_3 \geq 0. \end{aligned} \tag{5.34}$$

Inspection reveals that the integer solution to this problem is $x_1^* = 1$, $x_2^* = 1$, $x_3^* = 0$, $z^* = 9$. Similar perusal of the problem also shows that (a) omission of the upper bound on x_1, [constraint (5.34b)] yields the solution $x_1 = 2$, $x_2 = 0$, $x_3 = 0$, $z = 10$; (b) omission of the upper bound on x_2 [constraint (5.34c)] yields the solution $x_1 = 0$, $x_2 = 2$, $x_3 = 1$, $z = 11$; and (c) omission of the upper bound on x_3 [constraint (5.34d)] yields the solution $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $z = 10$. Evidently, omission of any of the upper bounds results in a solution which differs from the

[33] That is, assuming that a computer code possibly using a different algorithm, may be employed to yield the integer optimum

integer solution of the complete problem. Hence none of these restrictions may be neglected, and it is not possible to utilize linear programming for obtaining the recomputed duals for the Lorie-Savage problem in general.[34]

The particular numerical example of Lorie and Savage of Table 2.1 does lend itself to these methods for obtaining the recomputed dual values. The same applies to the more interesting example, in which this problem is augmented by a tenth project, as was presented in Chapter 4, and which is repeated in Table 5.16. Examination of this table reveals

TABLE 5.16. THE AUGMENTED LORIE-SAVAGE NUMERICAL EXAMPLE

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}		Right side
b_j	14	17	17	15	40	12	14	10	12	15		
c_{1j}	12	54	6	6	30	6	48	36	18	6	$\leq$	50
c_{2j}	3	7	6	2	35	6	4	3	3	7	$\leq$	20

$(0 \leq x_j \leq 1)$

that upper bounds of unity on projects 2, 5, 7 and 8 are redundant. First period outlays for each of these projects exceed one-half the first period budget. The second period budget constraint does not make any additional upper bounds superfluous for similar reasons, the upper bound on project 5 having already been found unnecessary due to its first period outlay. The number of remaining explicit restrictions is now eight — the two budget constraints and the upper bounds on projects 1, 3, 4, 6, 9 and 10, while the number of original nonbasic variables is ten. Further examination of this example reveals that the upper bounds on projects 6, 9 and 10 are similarly not required for obtaining the integer solution, although nonnegativity constraints for projects 2, 5, 6, 7 and 8 are needed. These restrictions number ten, and it is possible to obtain the integer optimum from the linear programming optimum by the addition of a cutting plane at each iteration and without making nonbasic any of the variables which are basic at the linear programming optimum, as is shown in Table 5.17. Hence "backward pivoting" from the integer to the linear programming optimum is possible, and the recomputed duals are the duals of the linear programming solution. These are given in Table 5.18.

[34]Omission of the budget constraint (5.34a) shifts the integer optimum to $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $z = 12$. Hence the constraint set cannot be reduced by neglecting this restriction.

TABLE 5.17. INTEGER SOLUTION TO THE AUGMENTED LORIE-SAVAGE EXAMPLE[a]

Initial Tableau

	1	$-x_1$	$-x_2$	$-x_3$	$-x_4$	$-x_5$	$-x_6$	$-x_7$	$-x_8$	$-x_9$	$-x_{10}$
z =		−14	−17	−17	−15	−40	−12	−14	−10	−12	−15
r_1 =	50	12	54	6	6	30	6	48	36	18	6
r_2 =	20	3	7	6	2	35	6	4	3	3	7
$\bar{x}_1$ =	1	1									
$\bar{x}_3$ =	1			1							
$\bar{x}_4$ =	1				1						

[a]Upper bounds on x_2, x_5, x_6, x_7, x_8, x_9 and x_{10} are omitted. Iterations between the initial and the linear programming optimal tableau have been left out to conserve space. Problems of round-off error have been avoided by use of fractions in place of decimals.

L. P. Optimal Tableau

	1	$-r_1$	$-r_2$	$-\bar{x}_1$	$-\bar{x}_3$	$-\bar{x}_4$	$-x_2$	$-x_5$	$-x_6$	$-x_7$	$-x_8$
→ z =	71 8/9	13/36	11/6	25/6	23/6	20/3	46/3	35	7/6	32/3	17/2
x_9 =	32/27	7/108	−1/18	−11/18	−1/18	−5/18	28/9		1/18	26/9	13/6
x_{10} =	7/9	−1/36	1/6	−1/6	−5/6	−1/6	−1/3	5	5/6	−2/3	−1/2
x_1 =	1			1							
x_3 =	1				1						
x_4 =	1					1					
s_1 =	−8/9	−13/36*	−5/6	−1/6	−5/6	−2/3	−1/3		−1/6	−2/3	−1/2

TABLE 5.17 (cont.). INTEGER SOLUTION TO THE AUGMENTED LORIE-SAVAGE EXAMPLE[a]

Tableau 3

	1	$-s_1$	$-r_2$	$-\bar{x}_1$	$-\bar{x}_3$	$-\bar{x}_4$	$-x_2$	$-x_5$	$-x_6$	$-x_7$	$-x_8$
z =	71	1	1	4	3	6	15	35	1	10	8
x_9 =	40/39	7/39	−8/39	−25/39	−8/39	−31/78	119/39		1/39	36/13	27/13
→ x_{10} =	11/13	−1/13	3/13	−2/13	−10/13	−3/26	−4/13	5	11/13	−8/13	−6/13
x_1 =	1			1							
x_3 =	1				1						
x_4 =	1					1					
r_1 =	32/13	−36/13	30/13	6/13	30/13	24/13	12/13		6/13	24/13	18/13
s_2 =	−11/13	−12/13*	−3/13	−11/13	−3/13	−23/26	−9/13		−11/13	−5/13	−7/13

Tableau 4

	1	$-s_2$	$-r_2$	$-\bar{x}_1$	$-\bar{x}_3$	$-\bar{x}_4$	$-x_2$	$-x_5$	$-x_6$	$-x_7$	$-x_8$
→ z =	70 1/12	13/12	3/4	37/12	11/4	121/24	57/4	35	1/12	115/12	89/12
x_9 =	31/36	7/36	−1/4	29/36	−1/4	−41/72	35/12		−5/36	97/36	71/36
x_{10} =	11/12	−1/12	1/4	−1/12	−3/4	−1/24	−1/4	5	11/12	−7/12	−5/12
x_1 =	1			1							
x_3 =	1				1						
x_4 =	1					1					
r_1 =	5	−3	3	3	3	9/2	3		3	3	3
s_1 =	11/12	−13/12	1/4	11/12	1/4	23/24	3/4		11/12	5/12	7/12
s_3 =	−1/12	−1/12	−3/4	−1/12	−3/4	−1/24	−1/4		−1/12*	−7/12	−5/12

TABLE 5.17 (cont.). INTEGER SOLUTION TO THE AUGMENTED LORIE-SAVAGE EXAMPLE[a]

Tableau 5

	1	$-s_2$	$-r_2$	$-\bar{x}_1$	$-\bar{x}_3$	$-\bar{x}_4$	$-x_2$	$-x_5$	$-s_3$	$-x_7$	$-x_8$
z =	70	1		3	2	5	14	35	1	9	7
x_9 =	1	1/3	1	34/36	1	−1/2	10/3		−5/3	11/3	8/3
x_{10} =		−1	−8	−1	−9	−1/2	−3	5	11	−7	−5
x_1 =	1			1							
x_3 =	1				1						
x_4 =	1					1					
r_1 =	2	−6	−24		−24	3	−6		36	−18	−12
x_6 =	1	1	9	1	9	1/2	3		−12	7	5

(Integer optimum)

TABLE 5.18.
RECOMPUTED DUAL VARIABLES FOR THE AUGMENTED LORIE-SAVAGE EXAMPLE

$\rho_1^* = 0.361$	$\mu_1^* = 4.167$	$\mu_4^* = 6.667$	$\gamma_5^* = 35.0$	$\gamma_7^* = 10.667$
$\rho_2^* = 1.833$	$\mu_3^* = 3.833$	$\gamma_2^* = 15.333$	$\gamma_6^* = 1.167$	$\gamma_8^* = 8.500$

At the linear programming optimum, as Table 5.17 reveals, the unity upper bound on project 9 is violated $x_9 = 32/27$. Nevertheless, at the integer optimum its value drops to one. As a result, however, the omitted upper bound constraint receives a zero recomputed dual value. Similarly, the tenth project appears in the linear programming solution at a positive level, though less than unity. Hence the implied dual value for both its upper bound constraint and its nonnegativity constraint is zero. We defer further comment on these numerical results until we may compare them to quantities obtained by a process different from the above but alternative to it, which we take up in the next section.

5.8. Alternatives to the Gomory and Baumol Recomputed Dual Prices

Having analyzed the properties of the recomputed duals of Gomory and Baumol and seen the difficulties which they entail, we shall formulate alternative quantities, and illustrate them in the context of the Lorie-Savage model.[35] Our alternate duals will be shown to possess many of the properties of the recomputed duals of Gomory and Baumol. They also have some additional properties, but lack others, as we shall presently indicate.

We may indicate the motivation for the approach we shall take by focussing once more on the Lorie-Savage method for solving this problem. We recall that their method requires finding quantities p_t with the properties,

$$\sum_t p_t c_{tj} \leq b_j \tag{5.35a}$$

for accepted projects,

$$\sum_t p_t c_{tj} > b_j \tag{5.35b}$$

for rejected projects, and

$$p_t \geq 0.^{36} \tag{5.35c}$$

[35]Although discussion of the alternate duals is here confined to the Lorie-Savage model, the approach is quite general. A broader discussion of it is inappropriate here and will be presented elsewhere.

[36]Lorie and Savage require the p_t to be strictly positive (*op. cit.*, p. 234).

The similarity between the Lorie-Savage auxiliary variables p_t and the dual variables of the linear programming model has already been discussed in Section 3.5. Since the solution which Lorie and Savage sought was one of integral projects, we may attempt to reverse the procedure they suggest and instead formulate a linear programming problem which yields as its solution the quantities p_t.

To accomplish this we first determine the set J of accepted projects by using the integer programming model (4.1). We next solve a *linear* programming problem whose constraint set is

$$\begin{aligned} \sum_t p_t c_{tj} &\geq b_j, \qquad j \in J \\ \sum_t p_t c_{tk} &\leq b_k, \qquad k \notin J \\ p_t &\geq 0. \end{aligned} \tag{5.36}$$

Values of p_t with properties (5.35) may be found for the original Lorie-Savage example; however, no values of p_t exist with these properties when the tenth project of Section 4.2 is used to augment the Lorie-Savage example. In this latter problem infeasibility is indicated.[37]

Infeasibility of the augmented Lorie-Savage example above is a consequence of the discreteness requirement, and because of this requirement it may be the case that no set of prices exists which makes the chosen activities profitless[38] and which also causes the unaccepted activities to incur losses[39] — should the latter be carried out. We have seen that the Gomory and Baumol recomputed dual prices sometimes assign positive values to the nonnegativity constraints, which in effect constitute subsidies to some accepted projects.[40]

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[37]This computation is proof (alternative to the geometric proof of Section 4.2) that the Lorie-Savage method will not necessarily work. Because of the infeasibility, the form of the functional used in the numerical calculations is of no interest.

[38]In the opportunity cost sense.

[39]Or, at best, remain profitless.

[40]"There are, in the discrete programming case, likely to arise efficient outputs which are not competitive outputs and which cannot be enforced by the standard type of decentralized control procedure of the economic literature, in which the central authority makes only simple price decisions." (Gomory and Baumol, *op. cit.*, p. 535.) A similar observation was made earlier by T. C. Koopmans and Martin Beckman in "Assignment Problems and the Location of Economic Activities," *Econometrica*, XXV, No. 1 (January, 1957), pp. 53-76. See also T. C. Koopmans and A. F. Bausch, "Mathematical Reasoning in Economics," *SIAM Review*, I, No. 1 (July, 1959), Topic 9, pp. 127-9.

We may clarify the economic significance of discreteness by reference to Fig. 5.4. The four quadrants of this figure are divided by a vertical line between accepted and rejected projects, and by a horizontal line between projects earning positive and projects earning negative goodwill.[41]

	Accepted Projects	Rejected Projects
$b_j - \sum_t \rho_t^* c_{tj} \geq 0$	Positive[a] Goodwill (a)	(b)
$b_j - \sum_t \rho_t^* c_{tj} \leq 0$	(c)	Negative[b] Goodwill (d)

[a]Nonnegative. [b]Nonpositive.

Figure 5.4.

In the linear programming model without integer requirements it is a condition of optimality that all projects be classifiable into either of the two quadrants on the main diagonal. That is to say, projects either are accepted and earn nonnegative goodwill [quadrant (a)], or they are rejected and earn nonpositive goodwill [quadrant (d)].[42] The imposition of the integer requirement may make it necessary to utilize a third quadrant of the diagram for classifying projects — that is, subsidies may be required for projects falling in quadrant (c), or, alternatively, the positive goodwill of projects falling in quadrant (b) must be removed by a penalty analogous to taxing them.

The recomputed dual prices of Gomory and Baumol utilize all four quadrants of our figure. In Problem 5a (Table 5.13), we have observed that a rejected project may have an excess of present value over "cost," when evaluation of budget dollars utilized makes use of the recomputed dual prices.[43] Similarly, the augmented Lorie-Savage example illustrates that an accepted project may have a negative goodwill imputed to it, that is, that a "subsidy" is required to make it acceptable.[44] Clearly, in the presence of discrete inputs, two of the four quadrants of Fig. 5.4

[41]More precisely, between projects earning nonnegative and nonpositive goodwill.

[42]For accepted projects, $\mu_j^* \geq 0$, and for rejected projects, $\gamma_j^* \geq 0$.

[43]That is, $x_2^* = 0$, $\mu_2^* = \frac{1}{5}$.

[44]$x_6^* = 1$, $\gamma_6^* = \frac{7}{6}\left(\text{or,} \quad b_6 - \sum_t \rho_t c_{t6} = -\frac{7}{6}\right)$.

may be insufficient. At the same time, it is not necessary to utilize all four quadrants: it is possible to obtain evaluators of the scarce resources which make it possible to assign projects to quadrants (a) and (d), and *either* quadrant (b) *or* quadrant (d), but not to both of the latter.

One can devise quantities with the properties indicated above and at the same time achieve a restriction on the number of such objects by minimizing a linear function of them. This function arises in a natural manner. From analysis of the Lorie-Savage problem, it is natural to formulate a linear programming problem which, by construction, either allows positive goodwill only for accepted projects, or it allows negative goodwill only for rejected projects. The latter choice comes closer to the Gomory-Baumol formulation, and we consider it first.

Given the set J of projects which are accepted in the integer programming solution, we have identified all projects which may require a "subsidy" in the sense of Gomory and Baumol. The following linear programming problem then yields the desired quantities directly, the integer programming problem being assumed to have been solved by any of the available algorithms or computer codes.

$$\begin{aligned}
&\text{Minimize} && \sum_{t=1}^{T} \rho_t C_t + \sum_{j \in J} \mu_j \\
&\text{Subject to (a)} && \sum_{t=1}^{T} \rho_t c_{tj} + \mu_j - \gamma_j \geq b_j, \qquad j \in J \\
&\text{(b)} && \sum_{t=1}^{T} \rho_t c_{tk} \geq b_k, \qquad k \notin J \\
&\text{(c)} && \rho_t, \mu_j, \gamma_j \geq 0.
\end{aligned} \tag{5.37}$$

In this formulation the goodwill of accepted projects is expressed in (5.37a) as the difference between two nonnegative variables, which implies that it may be either positive or negative. If positive, we have excess over "cost"; if negative, the difference is the amount of the "subsidy" which is required to make the project "break even." Restriction (5.37b), however, makes it impossible for a rejected project to receive positive goodwill. Placing a functional on this problem reduces the number of such quantities ρ_t, μ_j and γ_j to a single set, assuming there are no alternate optimal solutions. The form of the functional can best be explained by examining the dual to (5.37).

$$
\begin{aligned}
&\text{Maximize} && \sum_{j\in J} b_j x_j + \sum_{k\notin J} b_k x_k \\
&\text{Subject to (a)} && \sum_{j\in J} c_{tj} x_j + \sum_{k\notin J} c_{tk} x_k \le C_t, \qquad t = 1, \ldots, T \\
&\text{(b)} && x_j \le 1, \qquad j \in J \\
&\text{(c)} && -x_j \le 0, \qquad j \in J \qquad \text{(that is, } x_j \ge 0\text{)} \\
&\text{(d)} && x_k \ge 0.
\end{aligned} \tag{5.38}
$$

This maximization problem is the original integer programming problem in which the upper bounds on the rejected projects have been eliminated. While these upper bounds may be needed to obtain the integer optimum, at the integer optimum they will not be binding.[45]

To remove the need for subsidies on accepted projects, we may modify model (5.37) in the following way, although by doing so we permit penalties to be placed on some rejected projects to eliminate the possibility of "profits" on these.[46]

$$
\begin{aligned}
&\text{Minimize} && \sum_{t=1}^{T} \rho_t C_t + \sum_{j\in J} \mu_j \\
&\text{Subject to (a)} && \sum_{t=1}^{T} \rho_t c_{tj} + \mu_j \ge b_j, \qquad j \in J \\
&\text{(b)} && \sum_{t=1}^{T} \rho_t c_{tk} + \mu_k - \gamma_k \ge b_k, \qquad k \notin J \\
&\text{(c)} && \rho_t,\ \mu_j,\ \mu_k,\ \gamma_k \ge 0.
\end{aligned} \tag{5.39}
$$

This problem also has a dual, which we may exhibit as follows:

$$
\begin{aligned}
&\text{Maximize} && \sum_{j\in J} b_j x_j + \sum_{k\notin J} b_k x_k \\
&\text{Subject to (a)} && \sum_{j\in J} c_{tj} x_j + \sum_{k\notin J} c_{tk} x_k \le C_t, \qquad t = 1, \ldots, T \\
&\text{(b)} && x_j \le 1, \qquad j \in J \\
&\text{(c)} && x_k \le 0, \qquad k \notin J \\
&\text{(d)} && -x_k \le 0, \qquad k \notin J.
\end{aligned} \tag{5.40}
$$

In effect, for purposes of obtaining suitable quantities to serve as evaluators of the budget constraints in the Lorie-Savage type model,

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[45] See Problem 6.

[46] That is, we make use of quadrant (b) as well as quadrants (a) and (d) in Fig. 5.4.

restrictions (5.40c and d) together have removed the projects which have been slated for rejection by the integer programming solution. After the evaluators of the budget constraints have been obtained by use of model (5.39) or (5.40), the rejected projects can also be evaluated by computing

$$b_k - \sum_{t=1}^{T} \rho_t c_{tk} \tag{5.41}$$

as a side calculation. However, this value can be positive, negative or zero.

The solution to system (5.39), which we shall call the "alternate" dual values, has a number of interesting properties. First, it must be admitted that an alternate dual of zero does not indicate that the corresponding constraint would never be binding, no matter how much its constant term were increased. However, we have seen that the recomputed dual values of Gomory and Baumol signal the presence of redundant constraints only in relation to the other constraints (and the functional) actually being utilized to obtain the integer solution. In particular, the difficulty of identifying those upper bound constraints which are redundant given only the budget constraints is considerable, as inspection of the augmented Lorie-Savage example given in Table 5.17 will bear out.

Our alternate dual variables have an interesting property which we wish to contrast with the comparable situation in the recomputed duals of Gomory and Baumol. For the "unique" recomputed dual prices, that is, those obtained by use of the linear programming model which makes use of only n original constraints which alone would yield the integer optimum, equation (5.27) may be rewritten as follows:

$$z^* = \sum_{j=1}^{n} a_{0,j} x_j^* \leq \sum_{i=1}^{n} \pi_i^* a_{i,0} \tag{5.42}$$

where the $a_{0,j}$ are the cost coefficients, the π_i^* are the recomputed dual variables, and the $a_{i,0}$ are the right side terms of the original restrictions. This relation may be established by reference to the fact that the right side of inequality (5.42) is the value of the optimum solution to a linear programming problem, which we may call z_{LP}^*. This problem is the same as the integer programming problem before the first cutting plane is introduced. Hence, it is a less constrained problem than the integer

programming one, and its optimal value must be no smaller than the optimal integer solution given by z^* in (5.42).[47]

By contrast, our alternate duals do satisfy the following duality relation:

$$\sum_{j=1}^{n} b_j x_j^* = \sum_{t=1}^{T} \rho_t^* C_t + \sum_{j \in J} \mu_j^* \tag{5.44}$$

since restrictions (5.38c and d) guarantee that $x_k^* = 0$ for all $k \notin J$. This relation implies that the alternate duals evaluate the accepted projects in a way which is consistent with the total value of the investment program.

5.9. Application of the Alternate Duals to the Lorie-Savage Example

We may illustrate these alternate duals by use of model (5.39) with the modified Lorie-Savage example in which the tenth project has been included. The numerical solution to this model is presented in Table 5.19.

It may be observed parenthetically that these values are all integral and that the value of the functional, \$70, is the value of the optimal primal integer solution. Two of the dual prices are zero, the "premium interest rate" associated with use of first period budget dollars, ρ_1^*, and goodwill for project 6. In both the problem actually solved to obtain these values, using (5.39) or (5.40), and the original problem, given in Table 5.16, the unity upper bound on project 6 is not binding. Its

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[47]This result may be established more rigorously by reference to the last equation on page 549 in Gomory and Baumol (*op. cit.*). In that equation, the term $a_{0,0}^*$ represents the value of the integer solution, and the quantity in the first parentheses,

$$a_{0,0}^* + \sum_{i=1}^{n} \pi_i g_{i,0},$$

is the value of the linear programming solution. Since the π_i are nonnegative as a condition of optimality of the integer programming solution, and the $g_{i,0}$ are strictly positive because of their origin in the construction of the cutting planes, we must have

$$a_{0,0}^* \leq a_{0,0}^* + \sum_{i=1}^{n} \pi_i g_{i,0}. \tag{5.43}$$

In the case of our Lorie-Savage problems, even when uniqueness of the recomputed duals fails to obtain, the relation (5.43) holds. As indicated earlier, lack of uniqueness in the recomputed duals here is due to the presence of degeneracy — more than one subset of the original set of constraints yields the integer optimum solution.

TABLE 5.19. ALTERNATE DUAL EVALUATORS FOR THE AUGMENTED LORIE-SAVAGE EXAMPLE

	$\rho_1^* = 0$	$\rho_2^* = 2$	$z^* = \$70$	
$\mu_1^* = 8$	$\mu_3^* = 5$	$\gamma_5^* = 30$	$\mu_7^* = 6$	$\mu_9^* = 6$
$\mu_2^* = 3$	$\mu_4^* = 11$	$\mu_6^* = 0$	$\mu_8^* = 4$	$\mu_{10}^* = 1$

Accepted projects are in italics.

omission would not alter either the linear programming solution or the integer solution. The same cannot be said for the zero value for ρ_1^*. When the projects rejected by the integer solution are omitted from consideration the first period budget constraint is not binding, a sum of \$2 being left over. However, the integer solution would be altered if this constraint were omitted from the original problem.

The positive goodwill indicated for some of the rejected projects demonstrates the difficulty of decentralization of the investment decision here. If the rates of interest ρ_t^*, were announced, only one project would be eliminated by these alternate dual variables. The remainder would be required to be passed on by a central authority, at which time the "penalties" on the projects to be rejected would be imposed. A similar procedure applied to the recomputed duals presented in Table 5.18 would lead to adoption of all projects earning positive goodwill, although some rejected projects (in this instance only project 6) would be given a "subsidy" by the central authority.

5.10. Conclusion

In this chapter we have reviewed the Method of Integer Forms of Gomory for the solution of integer programming problems. We have examined in detail the process proposed by Gomory and Baumol for the distribution of dual values associated with the cutting planes to the original constraints of the problem. The properties of these recomputed duals were detailed and previously summarized at the end of Section 5.6.

A different approach was employed for the formulation of alternate dual evaluators of the constraints of Lorie-Savage problems. These alternate duals were seen to arise in a natural manner from the problem, being obtained from the solution of a linear programming problem based on the actual integer solution. Thus the alternate quantities could be obtained without knowledge of the cutting planes employed in the integer optimization, and when other algorithms such as the All-Integer algorithm are employed. The alternate duals are hence not dependent

on the choice of additional constraints and possess the degree of uniqueness shared by the duals of linear programs in general; that is, they are unique in the absence of degeneracy. They do not identify "free goods" by imputing a zero value to the corresponding constraint as do the recomputed duals. However, a duality relation [equation (5.44)] holds for them, and they assign subsidies only to accepted activities, or, alternatively, tax only rejected ones. As a result it is easier to interpret them in the context of the problem under consideration.

In summarizing the solution to the Lorie-Savage and related problems by integer programming in general, we have seen that the procedures for obtaining integer solutions were found to have considerable drawbacks, both in terms of the computations required and in the interpretaton of the solutions. But, in contrast, the Lorie-Savage problem and its further generalization was seen in preceding chapters to possess certain structural properties that could be employed with advantage. In particular, these properties are such that they generally admit of fairly close approximations to the integer conditions by recourse only to standard linear programming techniques. Doubtless, still further exploitation of these special structural features could be utilized.[48] For these reasons we shall emphasize the development of models for solution by linear programming in the remainder of this study, turning first, however, to a brief discussion of the nature and relevance of budgets.

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[48]For example, by developing sub-dual algorithms as explained in Charnes and Cooper, *Management Models and Industrial Application of Linear Programming, op. cit.*, or Charnes, Cooper and Miller, "Dyadic Programs and Sub-Dual Methods," *Naval Research Logistics Quarterly*, VIII, No. 1 (March, 1961), pp. 1–24.

APPENDIX 5A

Example of Solution by the All-Integer Algorithm

Problem 1b

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 2x_2 \\ \text{Subject to} \quad & 4x_1 + 3x_2 \leq 5, \\ & x_1 \leq 1, \quad x_2 \leq 1. \end{aligned}$$

This is Problem 1a solved here by use of the All-Integer integer programming algorithm. The notation is that used in the other examples except that the auxiliary variables follow the notation used by Gomory in the All-Integer algorithm. This algorithm requires that the initial matrix be in integers only, as does the Method of Integer Forms, but in contrast to the latter, it maintains this condition through all iterations. Because the All-Integer algorithm proceeds via Lemke's Dual Method, it also requires that the initial matrix be dual feasible (that is, primal optimal, or, having lexicographically positive column vectors). This is not the case for our problem, but it is easily arranged by means of a suitable change of variable. Therefore, let

$$y_j \equiv 1 - x_j$$

and the transformed problem becomes

$$\begin{aligned} \text{Maximize} \quad & 5 - 3y_1 - 2y_2 \\ \text{Subject to} \quad & -4y_1 - 3y_2 \leq -2 \\ & y_1 \leq 1, \quad y_2 \leq 1, \\ & y_1 \geq 0, \quad y_2 \geq 0. \end{aligned}$$

The algorithm proceeds by adding a cutting plane at each iteration rather than first obtaining the linear programming optimum. The cutting planes are so constructed that the pivot element is always -1. This has the effect of making each element of the subsequent tableau an integer since the previous tableau was all-integer.[1]

Tableau 1

		1	$-y_1$	$-y_2$
z	=	5	3	2
$\rightarrow \bar{y}_1$	=	-2	-4	-3
$\bar{y}_2$	=	1	1	
$\bar{y}_3$	=	1		1
y_1	=		-1	
y_2	=			-1
s_1	=	-1	-1	-1^*

Tableau 2

		1	$-y_1$	$-s_1$
z	=	3	1	2
$\bar{y}_1$	=	1	-1	-3
$\bar{y}_2$	=	1	1	
$\bar{y}_3$	=		-1	1
y_1	=		-1	
y_2	=	1	1	-1

Integer Optimum

The row for pivoting is indicated by the arrow at the negative constant term. The column for pivoting is $j_0 = 2$, the (lexicographically) smallest column having a negative entry in the pivot row. To determine the equation of the cutting plane we compute quantities $\mu_j = [\alpha_j/\alpha_{j_0}]$, where the brackets indicate "integer part of" and the α_j represent the first nonzero element of column j. Here $\mu_1 = [\frac{3}{2}] = 1$; $\mu_2 = [\frac{2}{2}] = 1$. Next we determine the ratios $\lambda_j = -a_{rj}/\mu_j$, where r is the index of the pivot row, for all $j \epsilon J$, and J is the set of columns with $a_{rj} < 0$. Here $\lambda_1 = \frac{4}{1} = 4$; $\lambda_2 = \frac{3}{1} = 3$. Finally, we obtain $\lambda = \max_{j \epsilon J} (\lambda_j) = \max (\lambda_1, \lambda_2) = \max (4,3) = 4$, and we write the coefficients of the additional constraint as $[a_{rj}/\lambda]$. These have been entered in the s_1 row of Tableau 1. The pivot element is indicated by the asterisk. Tableau 2 is obtained by use of the Dual algorithm, and it is seen to be both primal and dual feasible, hence optimal.

The constraint given by the cutting plane is

$$s_1 \geq 0$$

or

$$-1 + y_1 + y_2 \geq 0,$$

that is,

$$y_1 + y_2 \geq 1$$

or

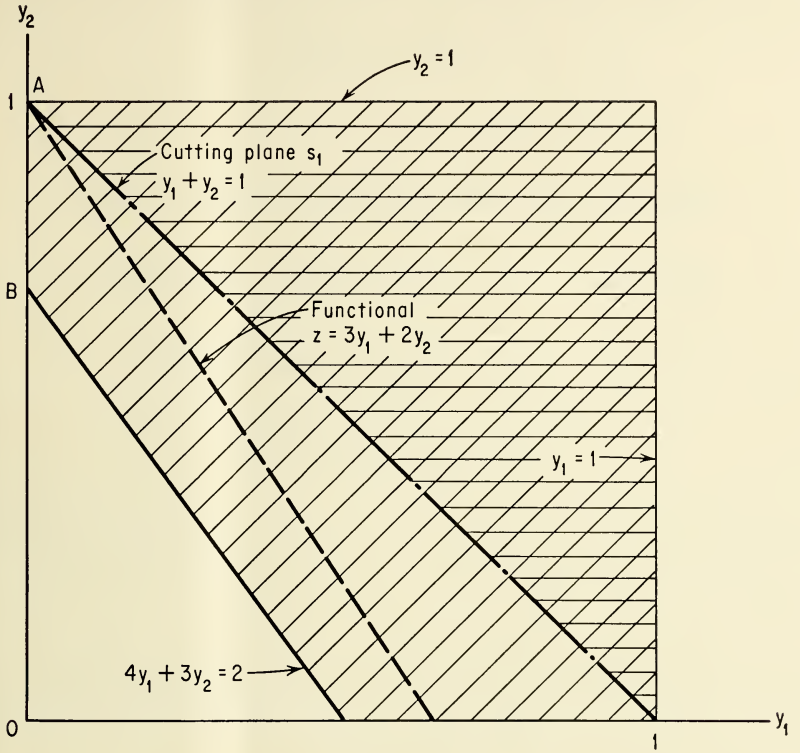
$$x_1 + x_2 \leq 1.$$

---

[1] A detailed statement of the algorithm requires more space than the present example justifies. It may be found in R. E. Gomory, "All-Integer Integer Programming Algorithm," *op. cit.*

The optimum is $y_1^* = 0$, $y_2^* = 1$, $\bar{y}_1 = \bar{y}_2 = 1$, $\bar{y}_3 = 0$, $z^* = 3$ which, in terms of the original variables, is $x_1^* = 1$, $x_2^* = 0$, $\bar{x}_1^* = \bar{x}_3^* = 1$, $\bar{x}_2^* = 0$, and $z^* = 3$.

A geometric representation of this solution is presented in Fig. 5A.1.



A = Integer optimal solution
B = L. P. Optimal solution (never appears)

[The original feasible region is diagonally shaded. The final feasible region is cross-hatched.]

Figure 5A.1.

APPENDIX 5B

Geometric Representation of the Integer Programming Problem and Some Implications for Decentralization

The following geometric formulation of the integer problem shows clearly that no single set of prices associated with each investment alternative can measure the effect on the total value when some project becomes unavailable, or when the budget for some period is changed.

In Fig. 5B.1 a two-period problem is represented by a set of rectangular blocks and a large rectangular box. The blocks, which symbolize the projects, have point values associated with them, and the problem consists of fitting blocks with as high a total point score within the box whose dimensions represent the budget constraints, with the requirement that (a) the blocks be "oriented" — have their horizontal edges always in a horizontal position and their vertical edges always vertical — and (b) the blocks be placed vertex to vertex, as in the diagram.

In the process of placing these blocks it may either be impossible or undesirable, in view of the total "point score," to reach the upper right corner of the box. If a block must be removed for some reason, it is quite likely that some other blocks already in place will also have to be replaced in order to achieve the maximum point score. The same may be required when the box is either reduced in size or enlarged.

To illustrate the amount of rearrangement necessary when budgets are

changed we may use the following illustration. If one were to plot the total value of investments against the amount of the budget in a one-period problem, as shown in Fig. 5B.2, the graph would always show discrete jumps. These jumps are not only due to the addition of projects, but also to the replacement of some projects by others, followed perhaps by their reciprocal replacement at a later stage. Hence, in programming language, the movement is not simply

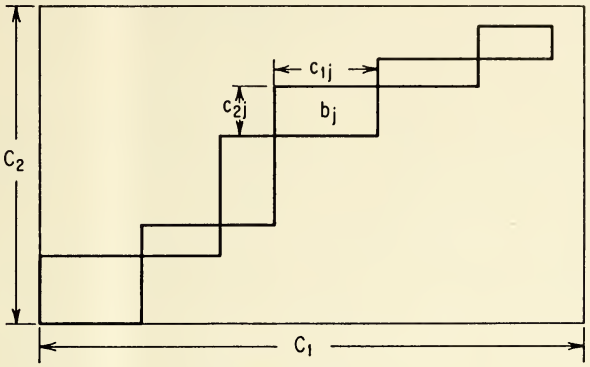


Figure 5B.1.

from one extreme point to the next, and parametric methods, to be discussed in Chapter 7 in conjunction with linear programming models, are far less efficient.

The graph below is derived from the original Lorie and Savage example[1] in which only second period outlays are considered. The budget is permitted to

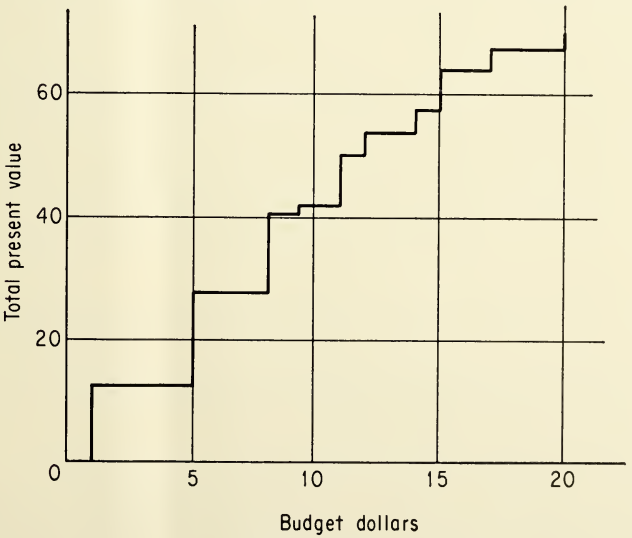


Figure 5B.2.

[1]Table 2.1, p. 9.

increase from zero to $20. Changes in the optimal program are given in Table 5B.1.

TABLE 5B.1. OPTIMAL INTEGER PROGRAMS FOR A ONE-PERIOD PROBLEM WHEN THE BUDGET IS EXPANDED[a]

Budget	Projects	Total Value	Projects Added	Projects Removed
$2	4	$15	4	
5	1, 4	29	1	
8	1, 4, 9	41	9	
9	1, 4, 7	43	7	9
11	1, 4, 8, 9	51	8, 9	7
12	1, 4, 7, 9	55	7	8
14	1, 3, 4, 9	58	3	7
15	1, 4, 7, 8, 9	65	7, 8	3
17	1, 3, 4, 8, 9	68	3, 7	7
20	1, 3, 4, 6, 9	70	6	8

[a]Based on second period outlays of the projects in Table 2.1.

CHAPTER 6

Some Comments on the Use of Capital Budgets

The term "budget" in capital budgeting shares some of the ambiguity surrounding use of this term elsewhere in the business literature. Two common uses of the term imply the diverse roles of budgets as *plans* and *controls*.[1] In the Lorie-Savage problem of allocating fixed dollar amounts among competing investment opportunities the term "budget" is used in the sense of a control over a firm's expenditures. The literature on capital budgeting does not often indicate in which sense the term is being used. In this chapter we shall discuss some of the evidence for the use of capital control budgets in the sense of the Lorie-Savage problem. We shall also comment briefly on the rationale for capital budgeting in the planning sense, drawing on interviews we have had with corporation executives. Our aim in the latter discussion is also to indicate the background for the models of the chapters which follow.

6.1. Capital Control Budgets in Business Practice

The motivation for the Lorie-Savage article was an actual situation confronting the authors while consulting for a large corporation.[2] More

[1]See A. C. Stedry, *Budget Control and Cost Behavior* (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1960), pp. 3–5.

[2]Reprinting of the article in *The Management of Corporate Capital* indicates that the problem still arouses interest and warrants attention.

recently, a doctoral dissertation by Donald Istvan[3] reports the results of detailed interviews with officials of 48 large corporations selected from the *Fortune Directory* of July 1957. These interviews were conducted during 1958, and concerned all aspects of the capital expenditure programs of the firms surveyed.

He discusses the use of budgets as follows:

> The short-range capital budget is a tool of management used to correlate the dollar amount of expenditure proposals with the funds available, or needed, for implementation of those proposals over a period of the next year, or in some instances, the next two years. . . . Where the funds available for capital expenditure projects are limited in amount to depreciation charges and retained earnings, the capital budgeting of the firms is usually concerned with the rationing of the limited funds among the best proposals available. The expenditure proposals submitted must compete among themselves for the limited funds, and the short-range capital budget is a device for indicating in advance the probable allocation of a given amount of expenditure. This type of short-range budget can be called a "rationing budget."[4]

Later on, in reporting on his findings Istvan says:

> The forms of rationing type capital budget in use vary because of differing individual needs among the firms. Twenty-nine of the 34 firms [which use a rationing type budget] have a one-year (next-year) budget while five include the next two years in short-range budgeting. The pattern for budget formation is generally as follows:
>
> 1. The financial officer of the corporation determines the total amount of corporate funds which will be available for capital-expenditures during the budget period. This is usually accomplished from four to five months prior to the end of the current period.
>
> 2. About three months prior to the end of the current period, the various departments, plants and divisions of the corporation

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[3]D. F. Istvan, "The Capital-Expenditure Decision-Making Process in Forty-Eight Large Corporations," (Unpublished D.B.A. dissertation, School of Business, Indiana University, August, 1959).

[4]*Ibid.*, p. 62.

submit the proposals in conceptual form which they would like to have implemented in the future period along with an estimated dollar cost of these proposals.

3. The financial officer and his staff then consider the conceptual proposals of the various divisions of the corporation and attempt to match them to the funds which will be available for the budgeted "year."[5]

While in most of these firms the budgeting phase is followed up by an additional appropriation stage, Istvan also reports, ". . . six firms go so far as to make the short-range budget completely concrete. In these six firms, the proposals submitted for budgeting purposes are in a completed state of analysis and inclusion in the budget is the same as final approval of the proposal."[6]

Less formal surveys of capital budgeting practices reveal ample corroborative evidence on the existence of spending ceilings in numbers of corporations. In "The Capital Expenditure Control Program, A Summary of Practice"[7] the observation is made that "For the most part, the possibilities for capital expenditures far outnumber the funds available for this purpose. One of the chief functions of the control program, therefore, is to see that expenditures stay within authorized limits." The same report states,

> The overall amount of the budget is generally determined in one of two ways. Under one method a maximum amount available for capital expenditures is decided at the executive level. The various review and coordinating groups then take steps to see that the budget presented for approval is in substantial agreement with this figure.[8]

Similarly, the National Industrial Conference Board in one of its surveys reports, "A few companies budget by first deciding what their total expenditures will be. They then allocate these funds throughout the organization."[9]

A prominent post-war study of Twin Cities manufacturing firms reported on by Walter Heller in "The Anatomy of Investment De-

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[5] *Ibid.*, p. 66.
[6] *Ibid.*, p. 67.
[7] *NAA Bulletin*, XL, No. 7 (March, 1959), Section 3.
[8] *Ibid.*, p. 10.
[9] "Controlling Capital Expenditures," *Studies in Business Policy*, No. 62 (New York: National Industrial Conference Board, Inc., April, 1953), p. 6.

cisions"[10] provides further evidence on this point, and also suggests some rationale. In commenting on the nature of the sample of firms in his study, Heller observes that managers tended also to be owners to a large extent. In this connection he comments:

> When managers' decisions to spend more on capital equipment reduce *their own* dividends as stockholders, the need for budget ceilings on such expenditures becomes less urgent — at least in so far as these ceilings are designed to assure a minimum dividend to appease stockholders.[11]

He also succinctly states the basis for the expenditure calculations:

> The financial budget (sometimes called the "appropriation") involves the working out of a dollars-and-cents ceiling for plant and equipment outlays. Such internal sources of funds as expected profits before taxes, provisions for depreciation, proceeds of sales of long-term investments and machinery, and decreases in working capital are totaled; to this sum may be added planned outside financing. From the gross figure are then subtracted such items as projected dividends, income taxes, long-term debt repayment, and inventory expansion, and other increases in working capital requirements. The remainder is then set up as the ceiling for capital expenditures.[12]

The empirical studies cited present evidence that the practice of allocating fixed dollar amounts among competing capital investments is not unusual. The Lorie-Savage problem concerns such allocations when expenditures on projects require more than one budget period and when budget ceilings exist for several periods. The evidence on the extent of this problem is more limited. First, the one-period problem may be all that some firms are able to deal with, both with respect to setting the spending ceiling and making the required allocation. At the same time, the importance of the Lorie-Savage problem is derived from the allocation of limited funds among projects having outlays in several periods. Only a smaller number of firms are concerned with such investments.[13]

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[10] *Harvard Business Review*, XXIX, No. 2 (March, 1951), pp. 95–103.

[11] *Ibid.*, p. 97.

[12] *Ibid.*

[13] Of course, the importance of the formal problem represented by the Lorie-Savage problem is derived from a variety of applications, a number of which will be discussed later on.

A study of M. Gort[14] of the electric power industry describes the nature of capital planning in one industry in which the multi-period problem exists. Although he reported no attempt at a formal solution to these problems, within this industry one may expect our models of the Lorie and Savage problem, and, perhaps more appropriately, the models of Chapters 8 and 9, to be most directly applicable.

Aside from long-range projections of demand for generating capacity which may extend beyond twenty years, and some ten year projections which might have implications for the present in terms of acquisition of rights-of-way, etc., Gort found:

> The planning of investment begins to acquire substantially greater importance at a point where it concerns itself with a period of approximately five years. This interval is still substantially longer than that normally covered by detailed capital expansion plans. However, certain elements of a five-year projection have a bearing on current decisions.[15]

He goes on:

> Such a forecast is intended to perform two functions. It is, in the first instance, designed to facilitate financial planning. That is, certain objectives are sought in the common-stock-preferred-bond ratios in the capital structure of the company. An indication of the total amount of money that is likely to be required over a period of years defines the limits which should be adhered to in the issuance of a particular class of securities at a given time. Second, a knowledge of future capacity requirements enables, on occasion, the installation of units larger than required by current capacity needs, but which are desirable inasmuch as they entail a smaller cost than the successive addition of small units.[16]

Gort also reports the existence of "the formally approved construction budget with a schedule of expected installation dates," which most commonly had expenditures mapped out over a period of three years. The pattern revealed by Gort's findings shows that the length of the horizon for construction budgets is directly related to the magnitude of

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[14]"The Planning of Investment: A Study of Capital Budgeting in the Electric Power Industry," *Journal of Business*, Part I, XXIV, No. 2 (April, 1951), pp. 79–95; and Part II, XXIV, No. 3 (July, 1951), pp. 181–202.

[15]*Ibid.*, pp. 81–82.

[16]*Ibid.*, p. 82.

the construction programs. One important reason for this is the length of manufacturing lead times of heavy generating equipment. We shall suggest an additional reason later, although Gort's Table 4[17] "Cumulative Expenditures of Turbogenerators at Five Month Intervals for 1936, 1939 and 1945 Authorizations" indicates that in one instance 45 months had elapsed between authorization of expenditure and 96.2 per cent completion, and four out of five times installation had not been completed 35 months after authorization, although contracts were signed immediately after their authorization. Considering the lead times associated with design and development or related activity, it would not be beyond reason for many types of firms to have meaningful horizons of five or more years.

6.2. Some Reasons for the Use of Capital Budgets

Although a detailed discussion on the nature and rationale for budgeting procedures is beyond the scope of the present undertaking, a few words on the use of budgets as planning devices may be in order. These comments are based in part on the "business literature,"[18] and partly on informal conversations with executives of several large corporations having responsibilities in the area of capital expenditure planning. The existence of budgets appears to be determined by organizational needs of the firms, explaining, in part, the diversity of practice found by Istvan. These organizational functions of budgets may be grouped roughly into three categories, at least for present purposes.

The first area in which formal budgets fulfill an administrative need is in the coordination of the effort required to arrive at capital expenditure decisions. Depending on the size of the outlays required, these decisions are made at various levels within the company structure. Great emphasis is placed by many companies on having as many proposals as possible ready for consideration at the same time. Side-by-side comparison of projects enables management to obtain a better grasp of the coherence of company plans, and to exercise a greater degree of uniformity in judging individual projects.[19]

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[17] *Ibid.*, p. 94.

[18] This includes, for example, publications of the National Association of Cost Accountants, the National Industrial Conference Board, *Harvard Business Review*, etc.

[19] See, for example, "The Capital Expenditure Program, A Summary of Practice," *op. cit.*

An equally important consideration in connection with the decision procedure is that scarce resources are utilized in the process of preparing plans and estimates for investment projects. While some decisions, such as replacing a part versus replacing the machine itself, are relatively simple, the larger investment projects involve considerable design and development work. Although "good" investment alternatives should always be considered as soon as they are ready, target dates for the completion of calculations are found useful in the administration of this type of research and planning work.

A second function served by capital budgets concerns the controller's department and the treasurer's office. This is the coordination of financial and physical plans. The presence of sizable transaction costs limits the number of transactions in the financial markets it pays firms to undertake.[20] Although bank loans and other devices may be used to smooth the discontinuities generated by fixed costs, their implications for cash flows and interest costs are conveniently analyzed in the form of periodic budgets. In addition, even bank loans have fixed costs associated with them.

Just as the development of investment alternatives requires use of scarce resources, specifically skilled manpower, so does financial planning. Thus it may be necessary also to limit the number of financial alternatives that are considered. Aside from the paperwork involved in making economic calculations, the required legal work is costly, and negotiations required to secure loans or to make other transactions also consume time.

Finally, once investment projects are accepted, a host of other plans must be made. Thus, our third rough, general category relates to post-decision activities. For example, where construction is carried out by company employees, scheduling of crews is required to avoid serious and costly swings in construction employment. Employee training, especially at the managerial level but also on the level of the worker, requires forward planning because of the long lead times involved. Similar planning is required where equipment lead times are long. Acceptance of a capital budget makes these planning tasks easier.

One additional point relates to the ubiquitous emphasis on the need

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[20]See J. B. Michaelson, "The Determinants of Dividend Policies" (Unpublished Ph. D. dissertation, Graduate School of Business, University of Chicago, September, 1961). Michaelson found greater variation in dividend payments among smaller firms than among larger ones, and also greater use of external financing for the larger firms, apparently for the reason that for smaller firms access to financial markets is more costly than increasing earnings retention by changing dividends.

to rank investment alternatives. As will be amplified in detail below[21] the usual decision criteria for accepting or rejecting capital projects[22] do not require rankings when perfect capital markets and independent projects are assumed and do not work in the presence of capital rationing or nonindependence. Either all projects with a positive present value are accepted, or all projects with a rate of return greater than some specified rate are accepted. The use to which ranks are put may, in part, be a holdover from days of postponing all but the most urgent investments. More likely, they reflect the rapidity with which circumstances change, so that a previously accepted project may have to be dropped and perhaps be replaced by "the next one in line," or an additional opportunity is discovered which will "bump" one of the accepted projects. When firms operate with decentralized divisions within which all but major capital expenditures are adopted without review by the central organization, providing for balance of investment programs between the divisions becomes an additional requirement of the budgeting process. For these reasons we shall continue to concern ourselves also with the problem of ranking investment alternatives.

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[21]Chapters 8 and 9.

[22]That is, the present value or the rate of return criteria.

CHAPTER 7

Programming Methods Applied to Related Problems

In the preceding two chapters we have taken a classic problem — that of allocating funds limited in each of a number of time periods among different investment alternatives having outflows in several periods — and shown it to be solvable by integer programming, and approximately solvable by linear programming. In those chapters the conditions of the problem were taken as fixed. The purpose of the present chapter is to extend the conditions of the original problem and thereby both to increase the realism and relevance of the problem and, in the process, also to demonstrate the versatility of programming techniques for dealing with this class of business decisions.

7.1. Budget Deferrals

Budgeting procedures are often adjusted to permit funds that remain unused in one budget period to be transferred to a later period. Such authorizations are frequently designed to discourage such practices as acquiring typewriters or other budgeted items at the end of a fiscal year merely for fear that subsequent budgets will be reduced by an unspent current surplus. Such practices must be controlled, of course, and while it is not entirely clear that procedures of deferral produce an efficient allocation of resources generally, it may nevertheless be useful to formulate and examine a model which corresponds to this situation.

In the simplest case — for example in a linear or integer programming formulation of the Lorie and Savage problem — permitting surplus funds of the first year to be used in the second, and surplus funds of the second year to be used in the third, etc., may be accomplished in various ways. One is a model constructed as follows. Let b_j, as before, represent the net present value of investment project j and c_{tj} represent the present value of the outlay required for the jth project in period t. Then let C_t be the present value of the budget ceiling for year t, and finally let s_t be the amount of unused funds of period t.

Apart from s_t, it will be observed, all of the variables are exactly the same as before. To reflect the transfer of funds between periods we can then introduce the variables s_t into our constraints as follows:

$$\begin{aligned}
&\text{Maximize} && \sum_{j=1}^{n} b_j x_j \\
&\text{Subject to (a)} && \sum_{j=1}^{n} c_{1j} x_j + s_1 = C_1 \\
&\text{(b)} && \sum_{j=1}^{n} c_{tj} x_j - s_{t-1} + s_t = C_t, \qquad t = 2, \ldots, T \\
&\text{(c)} && 0 \leq x_j \leq 1, \qquad j = 1, \ldots, n \qquad (7.1) \\
&\text{(d)} && s_t \geq 0, \qquad t = 1, \ldots, T.
\end{aligned}$$

To see what is involved here consider the first constraint, under (a). If in some solution we have

$$\sum_{j=1}^{n} c_{1j} x_j < C_1 \qquad (7.2)$$

then

$$s_1 = C_1 - \sum_{j=1}^{n} c_{1j} x_j > 0 \qquad (7.3)$$

represents the funds remaining after all capital budget expenditures are accounted for. But then by reference to the first constraint in the set (b) we have

$$\sum_{j=1}^{n} c_{2j} x_j + s_2 = C_2 + s_1 \qquad (7.4)$$

or

$$\sum_{j=1}^{n} c_{2j} x_j - s_1 + s_2 = C_2$$

so that s_1, the idle funds of period 1, are thereby shifted to period 2. Evidently we might also have a program in which

$$\sum_{j=1}^{n} c_{2j}x_j < C_2 + s_1, \tag{7.5}$$

in which event we have

$$s_2 = C_2 + s_1 - \sum_{j=1}^{n} c_{2j}x_j > 0 \tag{7.6}$$

for a shift to the third period constraint, and so on.

Observe now that no present worth factor is applied to any s_t so that, in each case, this amount represents simply idle cash. On the other hand, the desirability of leaving any cash idle is, for purposes of our model, considered relative to *all* relevant opportunities[1] and over all relevant periods. It follows that, to the extent that the model is valid, we are also considering the opportunity cost (compounded) of leaving any cash idle in one period so that it can be transferred to some other period and thereby increase the over-all return via

$$\sum_{j=1}^{n} b_j x_j.$$

The model allows a transfer of funds only from earlier to later periods. Transfers from later to earlier periods could also be included, but this would make little sense in the present context. Complete freedom with respect to timing of funds would rule out the need for budgets and would, in effect, collapse the whole problem to a one-period problem.[2]

7.2. Multiple Budgets

In practice, other conditions are often superimposed onto the financial budgets which have been considered to be the only limiting factors in capital planning up to this point. Heller, writing some years after the close of World War II, found, for example, that "the rate of capital expenditure had been, and still was being, held down to what the very scarce factor of 'brains' — engineering and managerial — could handle."[3]

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[1]For borrowing and lending externally see Chapters 8 and 9.

[2]In such a one-period model the coefficients would be

$$c'_j = \sum_t c_{tj}, \qquad C' = \sum_t C_t.$$

[3]*Op. cit.*, p. 102.

Some of the companies which were interviewed during the course of our own study expressed a similar concern over specialized manpower shortages. The availability of construction crews, and the company's ability to maintain construction activities at a reasonably constant level, were among the dimensions requiring consideration in planning capital expansion.

The difficulty considered least surmountable by direct means involves securing managerial personnel for newly created facilities or functions. In principle it may be possible to bid away skills of all kinds from competitors or from other firms. However, the companies interviewed or discussed in the literature felt it imperative that men receiving managerial responsibilities have sufficient tenure with their employers to absorb the company's way of doing things. In addition, in an expanding economy raiding competitors may be a costly way of staffing, particularly when training costs and delays are also taken into account. Other instances of "personnel budgets" arise very frequently in military planning, especially in the development of new weapons systems. A B-58 wing, for example, would be useless unless pilots could be recruited, trained and made ready in time to man the bombers as they were produced.

Personnel considerations of these types can readily be integrated with capital expenditure plans by the addition of restrictions, expressed also in terms of the capital projects. If d_j is the number of men (for example, men with a certain skill) required for project j, and if J is the set of projects requiring men with the specified skill, then if we let d represent the maximum number of such men that will be available, we can write

$$\sum_{j \epsilon J} d_j x_j \leq d.$$

This "manpower constraint" could then be adjoined to the other budgetary constraints in order to insure the feasibility of the capital expenditure program with respect to this limitation. (Where necessary, or desirable, time subscripts may also be added, as in

$$\sum_{j \epsilon J} d_{tj} x_j \leq d_t,$$

or more elaborate versions can also be introduced to allow for training, experience, and transfers over time.)

Similar additions to the purely financial budgets may also have to be imposed because of material limitations. One example cited to the

writer concerned the Bell Telephone System, whose member companies receive all their equipment of certain types from the Western Electric Company. The parent organization, The American Telephone and Telegraph Company, which acts as banker and financial coordinator for the companies, discovered that Western Electric was being "whipsawed," in the words of one member company executive. In some years, for example, member companies may have ordered mainly telephone instruments, while in other years dial equipment was in heavy demand. In order to avoid having to alter rates of production for various items so drastically, the parent organization began to exericse control over the demands placed on Western Electric. This was done by rationing the kinds of products that its subsidiaries could order from Western Electric.

Still another form of budgetary problem which has some applicability in practice concerns the employment of certain financial instruments backed by specific assets as collateral. Equipment trust certificates commonly used by railroads to purchase rolling stock are an example. Although these are issued expressly to obtain railroad cars, there is little doubt that they also free funds for other capital expenditures. Yet, additions to rolling stock must be coordinated with plans for other facilities. Thus it may be desirable to place equipment budgets into the same model with other capital expenditures, so that the contingency relations between them may be worked out in an efficient manner.

A form of supplementary budgeting is involved in the balance restrictions often imposed on the growth of a firm. For example, the directors of an integrated oil company are likely to be concerned not so much with individual projects as with the over-all relationships between the projects. Their targets relate to total expenditures on various functional segments of the enterprise such as marketing, production, drilling and exploration. While "grand total" budgets may exist for financial planning, there are also definite bounds placed on the shares to be allocated to these functional areas to maintain a balanced company structure.[4]

Other additional budgetary forms and procedures are represented by the widespread devices of authorizations or allotments. A multi-period capital expansion total may be authorized, for example, but with the additional requirement that the specific allotments also be reapproved before each period, thus creating what may be thought of as "budgets within budgets."

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[4]See, for example, Dean, *Capital Budgeting*, *op. cit.*, p. 9.

7.3. Dual Variables Associated with Multiple Budgets

Before leaving this class of problems — and we shall have more to say on it in another connection in a later chapter — it may be useful to exhibit the dual variables of some of these additional restrictions and thereby also show how these variables can be interpreted in such a context. For this we shall adjoin a "manpower constraint," of the kind already discussed, to a standard set of financial budget constraints.

In the simplest case, for example, if we add only one manpower constraint to a Lorie-Savage problem we have:

Primal

$$\begin{aligned} &\text{Maximize} && \sum_j b_j x_j \\ &\text{Subject to (a)} && \sum_j c_{tj} x_j \leq C_t, \qquad t = 1, \ldots, T \\ &\text{(b)} && \sum_j d_j x_j \leq d \\ &\text{(c)} && 0 \leq x_j \leq 1 \end{aligned} \tag{7.7}$$

Dual

$$\begin{aligned} &\text{Minimize} && \sum_t \rho_t C_t + \sum_j \mu_j + \nu d \\ &\text{Subject to (a)} && \sum_t \rho_t c_{tj} + \nu d_j + \mu_j \geq b_j, \qquad j = 1, \ldots, n \\ &\text{(b)} && \rho_t, \nu, \mu_j \geq 0. \end{aligned} \tag{7.8}$$

The dual variables are defined as in the earlier Lorie and Savage models, except that an extra variable, ν, is now required to evaluate the additional scarce resource of manpower. The dimension of the coefficient d_j is number of men (of a specified skill) for a given project, and d is the total number of men available for this type of work. Therefore, by dimensional analysis of the functional it is seen that ν must be in dollars per man, suitably adjusted for the time discount rates that apply to projects and funds.[5]

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[5]As is readily verified from the expression

$$\nu^* = \frac{1}{d}\left[\sum_{j=1}^{n} (b_j x_j^* - \mu_j^*) - \sum_{t=1}^{T} \rho_t^* C_t\right] \tag{7.9}$$

which holds when $d > 0$. (This comes from the relation between the primal and the dual functionals,

$$\sum_{j=1}^{n} b_j x_j^* = \sum_{t=1}^{T} \rho_t^* C_t + \sum_{j=1}^{n} \mu_j^* + \nu^* d, \tag{7.10}$$

which holds at an optimum.)

Another view of this same matter may be obtained as follows. For a project accepted by the optimal program we may again, as in equation (3.3), rewrite constraint (7.8a), as

$$\mu_j^* = b_j - \sum_t \rho_t^* c_{tj} - \nu^* d_j. \tag{7.11}$$

In contrast to the situation of the simple Lorie and Savage model of Chapter 3, an accepted project is now not exclusively evaluated by the discounted monetary quasi-rents that it generates, that is, by the difference between its net present value and the present value of the budget dollars utilized, as measured at their opportunity cost. Instead, if the manpower constraint is effective, the value of the required number of men in terms of foregone alternatives must also be taken into account. The last term in the expression for μ_j^* in (7.11) measures this. Since ν must be nonnegative and d_j is normally positive, an acceptable project must also pay this additional "premium" for the manpower use its acceptance implies.[6]

7.4. Parametric Methods

Another tool from mathematical programming that may be applied to budgeting problems is parametric programming. Parametric programming is the term applied to computational methods which are used to explore the solutions that result when certain parameters of a linear programming model are changed continuously, within a given range.[7]

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[6]It is also clearly possible that projects with positive financial goodwill may nevertheless have to be rejected because of their use of the scarce manpower factor, as can be seen from the fact that the expression for the value of the rejected projects here (corresponding to (3.6)) is

$$\gamma_j^* = \sum_t \rho_t^* c_{tj} - b_j + \nu^* d_j. \tag{7.12}$$

Hence there may be some Project k such that

$$\gamma_k^* > b_k - \sum_t \rho_t^* c_{tk} > 0 \tag{7.13}$$

when financial factors alone are considered.

[7]See, for example, T. L. Saaty and S. I. Gass, "The Parametric Objective Function, Part I," *Journal of the Operations Research Society of America*, II (1954), pp. 316–319; Part II, *ibid.*, III (1955), pp. 395–401; and S. I. Gass and T. L. Saaty, "The Computational Algorithm for the Parametric Objective Function," *Naval Research Logistics Quarterly*, II (1955), pp. 39–45. See also A. Charnes and W. W. Cooper, *Management Models*, *op. cit.*, Chapter IX, for a discussion of a different method — called the spiral method — which restricts the parameterization to subsets with designated properties — for example, efficient points only.

In particular, efficient methods have been developed and programmed for computers to solve two derivative linear programming problems. The first of these is frequently designated as "parametric cost row" while the second is "parametric right side."

To make the intended meaning of these assertions somewhat clearer let us revert to standard notation and consider *any* linear programming problem, namely,

$$\begin{aligned} &\text{Maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{Subject to} && \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, \ldots, m. \\ & && x_j \geq 0, \qquad j = 1, \ldots, n. \end{aligned} \tag{7.14}$$

To study variations in the cost row this may be replaced by a representation like the following:

$$\begin{aligned} &\text{Maximize} && \sum_{j} (c_j + \phi c_j') x_j \\ &\text{Subject to} && \sum_{j} a_{ij} x_j \leq b_i, \qquad i = 1, \ldots, m \\ & && x_j \geq 0. \end{aligned} \tag{7.15}$$

Here ϕ is a parameter that is allowed to vary in order to study the effects of changes in the coefficients c_j to any possible new sets of values

$$\hat{c}_j = c_j + \phi c_j.^{8} \tag{7.16}$$

The parametric right side (which is the parametric cost row of the dual problem) may also be depicted in like manner. Here a parameter θ is introduced into the constraints to secure expressions of the following form

$$\begin{aligned} &\text{Maximize} && \sum_{j} c_j x_j \\ &\text{Subject to} && \sum_{j} a_{ij} x_j \leq b_i + \theta b_i', \qquad i = 1, \ldots, m \\ & && x_j \geq 0 \end{aligned} \tag{7.17}$$

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[8]Usually variations on the parameter ϕ are only undertaken after an optimum has been achieved for the originally stated problem — that is, problem (7.15), with $\phi \equiv 0$. The procedures as coded for electronic computer usage are, of course, entirely automatic.

so that variations in an initially stipulated set of b_i's may be studied. As the parameter θ is allowed to vary, all optimal solutions for the corresponding linear programming problems are then obtained.[9] A print-out of the θ and ϕ ranges may, of course, be secured as desired, along with the program values for both the direct and dual problems. Then the likely ranges of validity for an actual application may be studied or new limitations be established on the b_i, and so on. These methods may be used to accomplish a variety of tasks, including, say, a general sensitivity analysis used to investigate how the programs may respond to all possible ranges of variation in ϕ or θ — for example, when the original c_j and b_i are in question — and also including more limited but specific variations in the original model.

We are here concerned only with those aspects which have a bearing on the original problem and the variants discussed in this chapter. Therefore, consider a parametric right side in which the primed coefficients of the right side are zero for the upper bound constraints, and equal to the original budgets for the T budget years. That is, consider

$$\begin{aligned} &\text{Maximize} \quad \sum_j b_j x_j \\ &\text{Subject to} \quad \sum_j c_{tj} x_j \leq C_t + \theta C_t, \qquad t = 1, \ldots, T \qquad (7.18) \\ &\qquad\qquad 0 \leq x_j \leq 1. \end{aligned}$$

Evidently both positive and negative variations in θ may be admitted in order to study, as desired, the effects of either increases or decreases in the budgetary amounts, C_t. As the value of θ is increased, for nonnegative θ, the resulting solutions are the programs corresponding to various proportional increases in the budgets. Setting all but one or a few of the coefficients which multiply θ to zero narrows the range of exploration down to proportional increases in budgets for those years only. For instance, if it were desired to discover the consequences of variations in the first year budget only while holding those for the other years at their initially fixed values, C_t, we would write only $C_1 + \theta C_1$ in the first expression and suppress θ in all of the other budgetary (financial)

[9]It should be mentioned that the unbounded set of solutions that corresponds to values of ϕ or θ are never (and need never be) stated explicitly because they are, in themselves, of little interest. Instead, a finite and usually relatively small number of solutions are stated to correspond to changes in the nonzero variables in the solutions. The value of the parameter ϕ or θ is given when this list changes, and the solution to any particular value of the parameter that is desired may be obtained by a process analogous to interpolation.

constraints. Alternatively, it is possible to vary only the financial budgets while the manpower limitations are maintained, or *vice versa*, and so on.

When a parametric right side procedure is used on the model with budget deferrals,[10] one interesting consequence follows. If only the first period budget is varied and all other budgets are initially set equal to zero — thus permitting funds requirements in later periods to be fulfilled by deferral of first period budgets as these are increased[11] — then the problem collapses to a single-period problem. This may be seen from the fact that all flows are stated in terms of present values and thus, in this instance, all outlays are derived from an enlargement of the first period budget. Hence all differentiation by timing of the flows is removed.

Parameterization may be viewed as a technique that adds considerable flexibility to linear programming methods for budgetary applications wherein many of the estimates may be of uncertain or dubious merit until their consequences are known. In particular these techniques permit a relaxation or a firming of many assumptions, as, for example, the specific budget ceilings in the Lorie and Savage models. We shall now show in the final section of this chapter how some of the coefficients associated with individual projects may be varied. By means of an example involving multiple budgets, we shall also show how certain nonlinear restrictions may be introduced.

7.5. Technical Substitution

In a recent article Charnes and Cooper[12] have shown how the usual techniques of sensitivity analysis may be extended so that variations in the coefficients of the matrix as well as its rims may also be studied in a systematic manner. They have also indicated how to interpret and extend this further to secure formulations that are immediately relevant to problems in which both operating and capital budgets are to be

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[10]See equations (7.1), above.

[11]That is, if restrictions (7.1a and b) are replaced by

$$\begin{aligned} &\text{(a)} \quad \sum_{j=1}^{n} c_{1j}x_j + s_1 = \theta C_1 \\ &\text{(b)} \quad \sum_{j=1}^{n} c_{tj}x_j - s_{t-1} + s_t = 0, \quad t = 2, \ldots, T. \end{aligned} \tag{7.19}$$

[12]See "Systems Evaluation and Repricing Theorems," *Management Science*, IX, No. 1 (October, 1962), pp. 33–49.

considered simultaneously. In brief, their formulation replaces the standard linear programming version

$$\begin{aligned} &\text{Maximize} \quad \sum_{j=1}^{n} c_j x_j \\ &\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, \ldots, m \\ &\qquad\qquad x_j \geq 0, \qquad j = 1, \ldots, n \end{aligned} \tag{7.20}$$

with the following general model:

$$\begin{aligned} &\text{Maximize} \quad \sum_{j=1}^{n} c_j x_j - \mathcal{G}_1(\ldots, \epsilon_{ij} x_j, \ldots) \\ &\qquad\qquad\qquad\qquad - \mathcal{G}_2(\ldots, y_i, \ldots) \\ &\text{Subject to (a)} \quad \sum_{j=1}^{n} (a_{ij} - \epsilon_{ij}) x_j \leq b_i + y_i \\ &\qquad\text{(b)} \quad \epsilon_{ij} x_j \leq \bar{\epsilon}_{ij} \bar{x}_j \\ &\qquad\text{(c)} \quad \sum_{j=1}^{n} c_j x_j - \mathcal{G}_1(\ldots, \epsilon_{ij} x_j, \ldots) \\ &\qquad\qquad\qquad\qquad - \mathcal{G}_2(\ldots, y_i, \ldots) \\ &\qquad\qquad\qquad \geq k \sum_{j=1}^{n} c_j x_j^* \\ &\qquad\text{(d)} \quad y_i, \epsilon_{ij}, x_j \geq 0. \end{aligned} \tag{7.21}$$

Then, as they observe, we may regard the y_i as additions to b_i purchased at a cost defined by the function $\mathcal{G}_2$ and, similarly, we may interpret the variables ϵ_{ij} as representing process improvements that give new coefficients $\hat{a}_{ij} = a_{ij} - \epsilon_{ij}$ which are obtainable via the expenditures prescribed by the $\mathcal{G}_1$ function. Finally,

$$\sum_{j=1}^{n} c_j x_j^*$$

represents the amount that can be optimally secured when all ϵ_{ij} and y_i are fixed at zero values. Interpreting the constant $k > 0$ as a suitable discount term, we consider only those variations in the ϵ_{ij} and y_i which will return at least as much as the best alternative investment outside the firm (or plant) will also yield.

This is evidently a problem involving nonlinear constraints as well as a nonlinear functional. Nevertheless, as the article indicates, it is

possible to achieve effective simplifications and suitable solution procedures when the functions δ_1 and δ_2 have suitable properties (for example, when they are convex and separable). Indeed, if the functions δ_1 and δ_2 are piecewise linear, as well as convex and increasing functions of their arguments, then it is possible to achieve an equivalent linear programming problem merely by replacing the original c_j coefficients with new $\hat{c}_j$ values in the functional.[13]

We do not propose to follow all of the ramifications needed to develop a facsimile of these procedures for our problem in capital budgeting. Rather, we shall content ourselves with a development that leads to an analogous formulation that is pertinent to some of the problems that we have already covered. This will be done in such a way that, at least implicitly, we can exhibit how the indicated extensions may be brought to bear on other problems that were not adequately covered in our discussion of parametric programming procedures.

In the formulation of the models of Section 7.2, limitations on factors other than the financial outlays were superimposed on the budget ceilings. In doing so, it was assumed that the outlays for individual projects were given, as were the manpower requirements.[14] It may be desirable to relax these assumptions and to permit, instead, the substitution of capital, in the form of increased outlays, for operating manpower. Alternatively, construction expense may be sutstituted for the use of construction manpower in given years, for example, by adding contract labor to own force account labor, or by effecting design changes that substitute monetary outlays for labor requirements, etc. We shall assume that the rate of substitution between funds and the additional scarce resource, here labor, is piecewise linear, and that the corresponding "isoquants" from the theory of production have the usual convex shape.

In Fig. 7.1 we display the design alternatives for a given project, that is, the curve of technical substitution between outlays on the project in a given year and the operating manpower which the project will require upon its completion.[15] The combinations of outlays and manpower are plotted with outlays (in present values) on the vertical axis and manpower requirements on the horizontal axis.

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[13]This is an example of what they refer to as a repricing theorem.

[14]This applies equally to construction manpower and to the manpower required to operate additional facilities.

[15]For example, a building may be designed to use elevators requiring human operators with a lower outlay during the year these are installed than the same building with automatic elevators. Similarly the "elevator starter" may be automated or not, with corresponding cost differences.

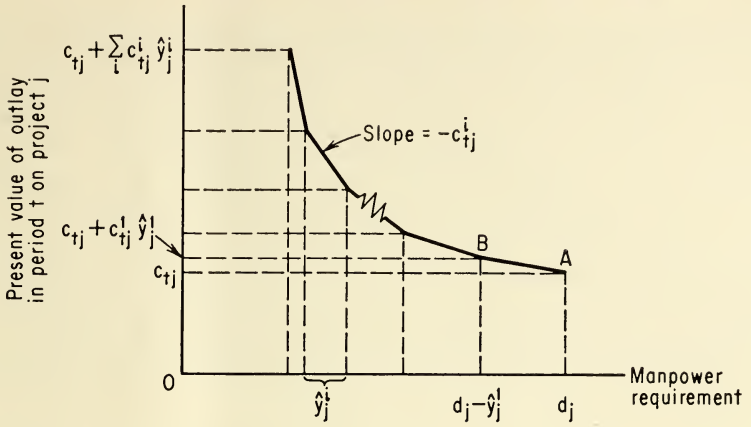

Figure 7.1.

We may describe how this curve was constructed by reference to a "base point" (point A in the diagram) at which the required outlay for year t is c_{tj} and the manpower requirement is d_j. We assume that point A depicts the maximum manpower requirement for the given project and that this point also corresponds to the point of lowest outlay.[16] We measure *changes* in the manpower requirements from the base requirement d_j and denote these by y_j. Thus, in moving along the first linear segment (from point A to point B in the diagram) the required outlays for Project j increase at the rate of c^1_{tj}. We denote the manpower change in this range by y^1_j, and also require that y^1_j be less than some upper bound $\hat{y}^1_j$. We similarly denote the rate of change in outlay corresponding to changes in the manpower requirement over the ith linear segment by $-c^i_{tj}$ and the corresponding manpower change by y^i_j, and require that $y^i_j \leq \hat{y}^i_j$. The concavity of the curve of substitution implies that $c^i_{tj} \leq c^{i+1}_{tj}$.[17] The total amount of manpower requirement substituted for by increased outlays on the project is then obtained by adding together the manpower requirement changes of the segments involved, that is, by

$$y_j = \sum_i y^i_j. \tag{7.22}$$

These changes in outlay due to alterations in the manpower requirement will also affect the net present value of the project, b_j in our

[16]That is, the required outlay for this amount of manpower is no lower at any point to the left of A.

[17]Since the slopes of the segments in the diagram are $-c^i_{tj}$.

notation. These effects may be portrayed as in Fig. 7.2, where the net present value of the project is plotted on the vertical axis, while the operating manpower is again shown on the horizontal axis. We again select as a base point (the point A in the diagram) the point at which the present value is b_j and the manpower requirement is d_j. Because a design for lower manpower requirement has higher outlays associated with it, one may assume that the corresponding present value of the project is also lower.[18] In moving from point A to point B in Fig. 7.2, the present value decreases at a rate of b_j^1, and becomes $b_j - b_j^1 \hat{y}_j^1$ at the point B. Over this segment, the decrease in manpower requirement is designated, as before, by y_j^1, to which the same upper bound $\hat{y}_j^1$ applies.[19] The other segments of the diagram may be described in similar terms, the

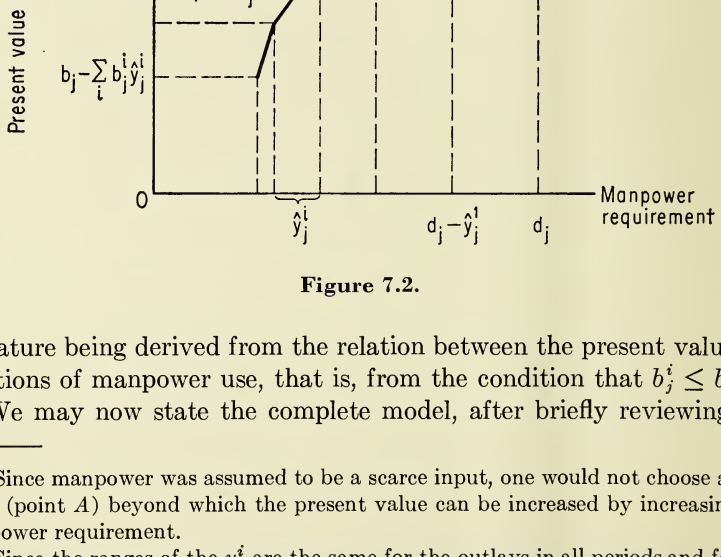

Figure 7.2.

curvature being derived from the relation between the present values as functions of manpower use, that is, from the condition that $b_j^i \leq b_j^{i+1}$.[20]

We may now state the complete model, after briefly reviewing the

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[18]Since manpower was assumed to be a scarce input, one would not choose a base point (point A) beyond which the present value can be increased by increasing the manpower requirement.

[19]Since the ranges of the y_j^i are the same for the outlays in all periods and for the present value function, the corners of Fig. 7.1 and 7.2 must have the same abscissas.

[20]It is possible that a reduction in manpower requirement may decrease the outlay for a project in some year while increasing it in others. This may mean that while the present value curve of Fig. 7.2 maintains its convexity, the curve of Fig. 7.1 for some year may no longer be concave throughout.

definitions of the remaining variables. As before, we denote by x_j the fraction of project j adopted; by C_t the present value of the budget ceiling for year t; and by d the total available manpower of the specified type.

$$\begin{aligned}
&\text{Maximize} && \sum_j b_j x_j - \sum_j \sum_i b_j^i y_j^i \\
&\text{Subject to (a)} && \sum_j c_{tj} x_j + \sum_j \sum_i c_{tj}^i y_j^i \leq C_t, \\
& && \qquad t = 1, \ldots, T \\
&\text{(b)} && \sum_j d_j x_j - \sum_j \sum_i y_j^i \leq d \qquad (7.23) \\
&\text{(c)} && y_j^i \leq \hat{y}_j^i x_j, \qquad i = 1, \ldots, m; \\
& && \qquad j = 1, \ldots, n \\
&\text{(d)} && x_j,\ y_j^i \geq 0.
\end{aligned}$$

We may note two points in regard to the nonlinearities of some of the restrictions in this model. First, the purpose of x_j in the upper bound restrictions (7.23c) is to change to zero the upper limit on the reduction of manpower associated with a project when the project in question is rejected. Without the addition of this factor it would be possible to accomplish a "reduction" in manpower use to fit the manpower constraint purely by reducing the employment of manpower in rejected proposals.[21] Second, because of the convexity of the present value function, as in Fig. 7.2, and the concavity of the substitution curve of Fig. 7.1, we need not be concerned with the possibility that $y_j^{*i+1} > 0$ when $y_j^{*i} < \hat{y}_j^i$.[22] This condition will not arise because the higher segments are more costly both in terms of the b_j^i of the functional and the c_{tj}^i of the budget constraints.[23]

7.6. Conclusion

In the present chapter we have developed a series of variations of models commonly overlooked in the traditional approaches to capital

[21] In this way we circumvent the implied nonlinearity of restrictions (7.23b) which might have been written as

$$\sum_j (d_j - \sum_i y_j^i) x_j \leq d,$$

thus involving products of variables.

[22] The asterisk again indicates the variable is at its optimal value.

[23] The situation described in footnote 20 page 136 may vitiate this result, in which case integer programming methods may be called for.

budgeting. These models have concentrated on the systematic exploration of alternatives related to changes in some of the initial conditions such as changes in budgetary constraints or in the imposition of such secondary conditions, as, for example, limitations in manpower. Making use of mathematical techniques that are both rigorous and computationally efficient, they enable a shift in emphasis from the evaluation of individual investment projects — which is all that prior approaches have attempted to accomplish — to an evaluation of an investment program as a whole.

The models presented in the chapter are by no means a complete catalogue of models which can be fruitfully applied to budgeting problems. Rather, they are intended to be suggestive of the breadth of analytic opportunity which the programming approach presents. Our next task will be the exploration of a different, though related, approach to the capital budgeting problem, one that accents the dynamic elements of capital planning.

CHAPTER 8

Alternative Formulation of Capital Budgeting Problems

Up to this point we have developed and discussed models of interest in themselves which serve to solve certain existing problems in the capital budgeting area in a computationally feasible and expeditious way. We shall next develop a class of models to clarify a number of issues in the economic literature and which will also (hopefully) prove to be somewhat more directly applicable to real world situations. The framework for this class of models is formulated in the present chapter in which we discuss some of the related issues and review a number of familiar results. In the following chapter this framework is applied to different situations of capital market imperfections.

Capital budgeting represents in some respects the central problem of the firm. The complexity of the problem derives from the fact that any set of actions taken today has consequences at later times, and the opportunities available at later dates are related to decisions being implemented currently. While the only decisions that need to be made today are those which require action today,[1] these decisions cannot ignore the range of opportunities at later times. Specifically, the decision to utilize resources for the acquisition of assets which yield streams

[1]See, for example, F. Modigliani and K. J. Cohen, "The Role of Anticipations and Plans in Economic Behavior and Their Use in Economic Analysis and Forecasting," *Studies in Business Expectations and Planning*, No. 4 (Urbana: Bureau of Economic and Business Research, University of Illinois, 1961).

of revenue but which cannot be turned back into cash or liquid assets without some cost calls for careful analysis. It is at this point that most of the traditional formulas for the economic evaluation of capital investment, such as the internal rate of return,[2] make one or more simplifying assumptions about the value of cash at future times — assumptions that may well be invalid and hence may lead to incorrect decisions.

One approach to these problems was first formulated and exploited by Charnes, Cooper and Miller[3] in an article which emphasized the interrelationships between the physical and financial flows within the firm and which demonstrated the usefulness of the linear programming apparatus for budgeting problems. We may analyze their contribution by contrasting it to that of Lorie and Savage.

The Lorie-Savage solution to the problem of allocating funds to investment opportunities under fixed spending ceilings involved two substantial difficulties. The first was setting the ceilings themselves, and the second was the choice of a discount rate for the purpose of calculating present values of the projects and of their outlays. Charnes, Cooper and Miller took a different tack. Instead of taking as data the discounted streams and the expenditure ceilings as did Lorie and Savage, they used the fundamental cost and revenue relations of the firm, leaving the elements of the stream, as well as the internal discount factors, to be determined by the model. In essence their model programs the entire set of economic activities of the firm.

The course we shall take here is intermediate between that of Lorie and Savage and that of Charnes, Cooper and Miller. While we shall emphasize those aspects relating to capital investment, we shall formulate our model so that some of the quantities which are inputs with Lorie and Savage are determined by the model simultaneously with the investment decisions. Instead of maximizing the present worth of the firm we will maximize its value as of some future terminal point which we call the horizon.[4] All flows will be current, and not present, values. The most important additional set of data will be the "cash throw-off" of the firm.

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[2]That rate of interest which makes the net present value of all revenues and costs equal to zero.

[3]A. Charnes, W. W. Cooper and M. H. Miller, "Application of Linear Programming to Financial Budgeting and the Costing of Funds," *Journal of Business*, XXXII, No. 1 (January, 1959), pp. 20–46.

[4]In this we again follow the example of Charnes, Cooper and Miller, *ibid.*, p. 3.

8.1. The Basic Horizon Model

Specifically, let a_{tj} be the flow in period t resulting from acceptance of project j.[5] The range of t will be from 1 to T, the horizon year. In contrast to the c_{tj} of the Lorie and Savage models, the a_{tj} may be either positive or negative since they represent all flows relating to a particular investment. We adopt the convention that a_{tj} is positive when it represents an expenditure and negative when it represents revenue. Since the stream of revenues associated with projects need not terminate at or before the horizon, we denote by $\hat{a}_j$ the value of all flows subsequent to the horizon, if any, discounted to the horizon at some rate of interest to be discussed presently.[6] Further, we denote by D_t the funds anticipated to be generated by the firm from operations in year t and which may be attributed to the resources the firm currently controls. Thus they are the anticipated future cash throw-off based on continuation of the firm, but excluding revenues to be derived from investments which the model is designed to determine.[7]

We introduce financial transactions into our model initially by means of lending and borrowing without limit at some stated rate of interest r, to be discussed more fully below. For the present we may assume this to be the market rate of interest.[8] We denote the amount borrowed in year t by w_t, while the amount available for lending, that is, carrying over to the next period at the same rate of interest is denoted by v_t. As in our earlier models, we let x_j represent the fraction of project j adopted. We then wish to maximize the net value of assets, financial and physical, as of the horizon, where the former are expressed in terms of the funds available for "lending" at that time, and the latter are represented by the discounted streams of net revenues past the horizon. We accomplish

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[5]This may be an increment in revenue attributable to the adoption of project j or, alternatively, an increment in cost.

[6]To simplify the model below, we also adopt the convention that $\hat{a}_j > 0$ denotes a positive asset value as of the horizon, while $\hat{a}_j < 0$ would represent a negative net value, for example, as associated with costly demolition at the end of the asset's life.

[7]For a discussion of investment horizons and funds horizons, see Friedrich and Vera Lutz, *The Theory of Investment of the Firm* (Princeton: Princeton University Press, 1951), pp. 25 ff.

[8]One may wish to add time subscripts to the interest rates to reflect differences in anticipated future rates which could occur even under the assumption of certainty.

this by means of the following linear programming model:

$$
\begin{aligned}
\text{Maximize} \quad & \sum_j \hat{a}_j x_j + v_T - w_T \\
\text{Subject to (a)} \quad & \sum_j a_{1j} x_j + v_1 - w_1 \leq D_1, \\
\text{(b)} \quad & \sum_j a_{tj} x_j - (1+r) v_{t-1} + v_t + (1+r) w_{t-1} \\
& \qquad - w_t \leq D_t, \qquad t = 2, \ldots, T \qquad (8.1) \\
\text{(c)} \quad & 0 \leq x_j \leq 1, \qquad j = 1, \ldots, n \\
\text{(d)} \quad & v_t, w_t \geq 0, \qquad t = 1, \ldots, T.
\end{aligned}
$$

Note that both lending and borrowing are accomplished by means of "renewable" one year contracts, where, for simplicity in writing, all interest is payable at the end of the year.[9] To the restrictions above we would add those expressing relationships of complementarity and competitiveness between projects, exactly as in our earlier models.

Solution of this model by means of mixed integer programming[10] with added restrictions that the x_j take on only integer values will result in the selection of the optimal set of investments, that is, those which maximize the present value of the firm, as will be shown in the next section. Solving the model by means of linear programming will permit the appearance of fractional projects in the optimal solution, with, as we shall see, analogous formal limitations and consequences to those of the earlier models, although without most of the drawbacks.

This linear programming horizon model, rather than the mixed integer version, is the basic model which will be contained in all the later variations. Before explaining these variations, however, it will perhaps be useful to show first that this terminal value or horizon model is exactly equivalent to maximizing the present value of the firm using the rate r as the discounting rate and that independent investment proposals are selected or rejected according to whether their present value is or is not nonnegative when computed at the same rate.[11] This

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[9]Since borrowing in the horizon year offers no opportunity for repayment, deduction of an amount equal to that borrowed in year T from the value of the functional avoids creation of a "money pump."

[10]Mixed integer programming rather than pure integer programming is indicated because the variables corresponding to amounts loaned or borrowed presumably need not be integral dollar amounts. Extension to the case of lump sum restrictions is straightforward.

[11]The case of nonindependent projects will be taken up later.

equivalence is best demonstrated by reference to the dual problem of the horizon model, to which we now turn.

8.2. The Dual of the Basic Horizon Model

Although their meaning is altered somewhat, there is some advantage in continuing with the symbols of the dual problems of previous chapters. The dual problem of the basic horizon model of the preceding section is:

$$\begin{aligned}
&\text{Minimize} && \sum_{t=1}^{T} \rho_t D_t + \sum_{j=1}^{n} \mu_j \\
&\text{Subject to (a)} && \sum_{t=1}^{T} \rho_t a_{tj} + \mu_j \geq \hat{a}_j, \qquad j = 1, \ldots, n \\
&\text{(b)} && \rho_T \geq 1 \\
&\text{(c)} && -\rho_T \geq -1 \\
&\text{(d)} && \rho_{t-1} - (1 + r)\rho_t \geq 0, \qquad t = 2, \ldots, T \\
&\text{(e)} && -\rho_{t-1} + (1 + r)\rho_t \geq 0, \qquad t = 2, \ldots, T \\
&\text{(f)} && \rho_t, \mu_j \geq 0.
\end{aligned} \tag{8.2}$$

Dual restrictions (8.2b and d) are derived from the lending activity of the primal problem, while restrictions (8.2c and e) come from the borrowing activities. The two conditions (8.2b) and (8.2c) written together are

$$1 \leq \rho_T \leq 1$$

which implies that

$$\rho_T = 1 \tag{8.3}$$

so that, also

$$\rho_T^* = 1. \tag{8.4}$$

Rewriting constraints (8.2d) and (8.2e) and combining them we may assert

$$1 + r \leq \frac{\rho_{t-1}^*}{\rho_t^*} \leq 1 + r, \qquad t = 2, \ldots, T \tag{8.5}$$

or

$$\frac{\rho_{t-1}^*}{\rho_t^*} = 1 + r. \tag{8.6}$$

Combining this with result (8.4) we obtain, for $t = 1, \ldots, T$,

$$\begin{aligned} \rho_t^* &= (1 + r)\rho_{t+1}^* = (1 + r)^2\rho_{t+2}^* = \ldots = (1 + r)^{T-t}\rho_{t+T-t}^* \\ &= (1 + r)^{T-t} \end{aligned} \tag{8.7}$$

since

$$\rho_{t+T-t}^* = \rho_T^* = 1.$$

The interpretation of the ρ_t^* is straightforward. They are incremental compounded rates of interest which express the yield at the horizon of an additional dollar in year t. The ratio ρ_t^*/ρ_{t+1}^* is the ratio of two compound interest rates, which is simply the annual incremental rate of interest. By equation (8.6), this is one plus r, the rate at which lending and borrowing take place.

Restrictions (8.2a) are understood most easily when transposed and with the reminder of the convention adopted in the preceding section that the a_{tj} are outflows when positive, inflows when negative. When a project is fully accepted, that is, $x_j^* = 1$, the corresponding restriction of the form of (8.2a) becomes an equation. Thus we obtain

$$\mu_j^* = \hat{a}_j - \sum_{t=1}^{T} \rho_t^* a_{tj}, \tag{8.8}$$

and $\mu_j^* \geq 0$ is the sum of the flows subsequent to the horizon plus those up to the horizon, when the sign of the flows is properly interpreted. Using equation (8.7), we may rewrite equation (8.8) as

$$\mu_j^* = \hat{a}_j + \sum_{t=1}^{T} (-a_{tj})(1 + r)^{T-t}. \tag{8.9}$$

However, recall that quantities $\hat{a}_j$ are the horizon values of flows subsequent to the horizon. We assume these also to be computed by discounting at the market rate r, that is, that

$$\hat{a}_j = \sum_{t=T+1}^{\infty} \frac{(-a_{tj})}{(1 + r)^{t-T}} \tag{8.10}$$

or,

$$\hat{a}_j = \sum_{t=T+1}^{\infty} (-a_{tj})(1 + r)^{T-t}.$$

Combining equations (8.9) and (8.10), we obtain[12]

$$\mu_j^* = \sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{T-t}. \tag{8.11}$$

We next divide both sides of equation (8.11) by the constant $(1 + r)^T$ to obtain

$$\frac{\mu_j^*}{(1 + r)^T} = \sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{-t}. \tag{8.12}$$

For simplicity, let

$$K \equiv (1 + r)^T.$$

This constant K is the compound accumulation factor up to the horizon date, T. For $j \in J$, the set of accepted projects, we then have

$$\mu_j^* = K \sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{-t}. \tag{8.13}$$

In words, μ_j^* is a constant times the present value of the sum of all flows associated with project j, when these are discounted at the rate r. The constant K transforms a present value to a horizon value. We may see exactly what this means by the following. Consider any term

$$\frac{a_{tj}}{(1 + r)^t}$$

in the range $1 \leq t \leq T$, say. Suppose that this a_{tj} is negative so that an inflow rather than an outflow is represented and recall that these are actual flows rather than present worths. But $a_{tj}/(1 + r)^t$ adjusts this a_{tj} to a corresponding *present* worth. If this present worth equivalent of a_{tj} were allowed to accumulate at rate r, we could have the amount

$$\frac{a_{tj}}{(1 + r)^t}(1 + r)^T$$

at the horizon date, T. Finally, to complete our exposition we observe that for $T \leq t \leq \infty$ we are also considering a present worth for the corresponding a_{tj}, and so exactly the same interpretation applies. Hence,

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[12]This series and also series (8.10) may be expected to converge. T is finite, so that for sufficiently large $t \geq t_K > T$ we have

$$\lim_{t \to \infty} \left| \frac{-a_{tj}}{(1 + r)^{t-T}} \right| < 1$$

for any $r > 0$ if only $a_{tj} \leq \bar{a}_{tj}$ for some fixed finite value of $\bar{a}_{tj}$.

via the constant K we are simply transforming all present values for receipts and expenditures to the relevant horizon date, T.

The terms $\rho_t^* D_t$ of the optimal functional in (8.2) are the values of the cash throw-off, compounded at rate r to the horizon. We may again utilize (8.7) and replace

$$\sum_{t=1}^{T} \rho_t^* D_t$$

by

$$\sum_{t=1}^{T} (1 + r)^{t-T} D_t = K \sum_{t=1}^{T} D_t (1 + r)^{-t} \tag{8.14}$$

and thereby also refer this part of the functional to our horizon date, T, in terms of the constant K times the *present* value of the cash throw-offs.

The remaining term

$$\sum_{j=1}^{n} \mu_j^*$$

in the dual functional also readily admits of this same adjustment, as may be immediately seen by inspecting (8.13) and observing that all $\mu_j^* = 0$ for the rejected projects. Hence, the value of the optimal dual functional is equal to the positive constant, K, times the present value of all streams of revenue from the chosen investments and from the existing facilities. We may call this the value of the firm.[13]

The value of the primal functional in (8.1) is also the horizon value of physical and financial assets of the firm. Since the model maximizes this value, and by the dual theorem the optimum value is equal to the optimum value of the dual functional, we see that we have maximized the present value of the firm.[14]

To show that all projects accepted by the program have nonnegative present values when computed at the interest rate r, and that no rejected project has a positive present value so computed, *when only independent projects are considered*, we need consider only two points. First, all

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[13]The value of the firm is usually defined as the discounted value of all net revenues. Here, by setting a finite horizon, we are in effect substituting purely financial opportunities, at the market rate of interest, for the opportunities for physical investment which may be available subsequent to the horizon. If T is chosen large enough, the difference is apt to be small. Similarly, if the firm is faced with a certain degree of competition, the value of its opportunities is unlikely to be too different from that of its competitors, especially in the distant future.

[14]See the last paragraph of Section 8.1.

variables in linear programming models must be nonnegative. Thus $\mu_j^* \geq 0$. We have seen that for the accepted projects the μ_j^* are the present values of the streams of revenues and costs associated with the projects multiplied by the positive constant K. Hence all accepted projects have nonnegative present values.

For the rejected projects we know that $x_j^* = 0$, and so also $x_j^* < 1$ and therefore $\mu_j^* = 0$.[15] But since restriction (8.2a) applies, in any event, for these projects we have

$$0 = \mu_j^* \geq \hat{a}_j - \sum_t \rho_t^* a_{tj}. \tag{8.15}$$

Via (8.7) and (8.10) we can then reduce this to

$$K \sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{-t} \leq 0, \tag{8.16}$$

where K, defined as before, converts a present to a horizon value. In words, the present value of a rejected project, computed at the rate r, is never positive.

8.3. The Effect of Contingency Relations

In the preceding section we have shown that in the basic model independent projects are never accepted unless their net present value at the borrowing-lending rate of interest is nonnegative and those with present values less than zero are always rejected. This conclusion must, of course, be modified for the case of interrelationships between projects, and so it may be well at this point to turn briefly aside from the main developments in order to clarify some of these subsidiary issues.

To see what is involved, let us first consider sets of mutually exclusive projects. The restriction expressing competitiveness is exactly the same as it was for the Lorie and Savage model of Chapter 3, that is,

$$\sum_{j \in J} x_j \leq 1 \tag{8.17}$$

for the set J of mutually exclusive projects. This constraint replaces the separate upper bounds of unity of the form of restrictions (8.1c) on the members of set J. If the optimal program of (8.1) rejects all projects

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[15]Projects accepted fractionally will have a net present value exactly equal to zero. For these, $\mu_j^* = 0$ since $0 < x_j < 1$, and restriction (8.2a) remains an equation. For this reason a fractional project may be eliminated from the optimal program without a decrease in the program value.

in this set, then it must be that the present value of each of them is zero or less. Here we may assert that if

$$\sum_{j \epsilon J} x_j^* < 1,$$

then $\mu_j^* = 0$ is a condition of optimality.

Replacing the individual unity upper bounds on the x_j by the condition that the sum of a certain number of them be no greater than one changes the interpretation of the dual variables somewhat. We can no longer identify the dual variable μ with an individual project, as was the case when separate restrictions of the form of (8.1c) were required.[16] If we regard the dual variables as evaluators of whole constraints of the primal problem, then we can label the dual variables that are of interest here with the subscript J that refers to the entire set of mutually exclusive projects.

Evidently $\mu_J^* \geq 0$ in any event. But μ_J is an opportunity cost. It is, in this sense, the cost of decreasing the number of projects that may be accepted from the set J. Because $\mu_J \geq 0$ enters with a positive coefficient, the dual functional in (8.2) will seek to minimize it, if possible, consistent with other aspects of the problem. Hence, for the accepted project from this set[17]

$$\mu_J^* = K \sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{-t} \tag{8.18}$$

while for the remaining members of set J,

$$\mu_J^* \geq K \sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{-t}. \tag{8.19}$$

To interpret this relation we may say that the present value of the accepted project is at least as high as that of any other member of the mutually exclusive set. This means, of course, that some *rejected* projects from the set may have positive present values when discounted at the rate r. Should the optimal program call for fractional acceptance of several projects within the same mutually exclusive set, the fraction of one project may be increased to unity and the other projects may be eliminated from the optimal solution. Justification for this step is

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[16]One could continue to include such restrictions, even though they would be redundant. However, the dual variables generated by the additional restrictions would not simplify this discussion.

[17]When several projects are accepted fractionally from among a single set of mutually exclusive projects, equation (8.18) holds for each of them.

related to the problem of the number of fractional projects, which we shall take up in the next section. In the meantime we must still concern ourselves with complementarity relationships.

Dealing with contingent projects[18] by constructing "compound" projects (as was shown in the first of the two methods of Section 3.6) amounts to their treatment as mutually exclusive projects, which we discussed above, and the same conclusion applies. The accepted compound project will have a nonnegative present value as large as that of any other compound project of the set.

The second method of expressing complementarities discussed in Section 3.6 took the form

$$x_f \leq x_g. \tag{8.20}$$

In this relation, acceptance of project f depends on acceptance of project g, although project g may be undertaken independently. This would be written as a restriction together with the essential upper bound on the independent alternative as

$$x_f - x_g \leq 0 \tag{8.21}$$

$$x_g \leq 1. \tag{8.22}$$

The corresponding dual constraints would take the form

$$\mu_k \geq \hat{a}_f - \sum_t \rho_t a_{tf} \tag{8.23}$$

and

$$\mu_g - \mu_k \geq \hat{a}_g - \sum_t \rho_t a_{tg} \tag{8.24}$$

where the subscript k has been chosen for the contingency restraint instead of using μ_f, as the latter would have been more closely identified with project f alone.

We take up the possibilities in order.

1. The independent project is accepted but the dependent one is rejected:

$$x_g^* = 1, \qquad x_f^* = 0.$$

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[18]For definitions, see p. 11.

From this we deduce that

$$\mu_g^* \geq 0, \qquad \mu_k^* = 0,$$

and

$$\begin{aligned} \mu_g^* &= \hat{a}_g - \sum_t \rho_t^* a_{tg} \qquad (8.25) \\ &= K \sum_{t=1}^{\infty} (-a_{tg})(1 + r)^{-t} \\ &= \text{Horizon Value of project } g \\ &= \text{Present Value of project } g \text{ (times } K) \\ &\geq 0. \end{aligned}$$

Also, the present value of the rejected project is nonpositive, that is, relations (8.15) and (8.16) apply.[19]

2. Both projects are accepted:

$$x_g^* = 1, \qquad x_f^* = 1.$$

There is no slack in either restriction (8.21) or (8.22), from which we may conclude that

$$\mu_g^* \geq 0, \qquad \mu_k^* \geq 0.$$

By relation (8.24), which must here be an equation, we may write

$$\begin{aligned} \mu_g^* - \mu_k^* &= K \sum_{t=1}^{\infty} (-a_{tg})(1 + r)^{-t} \qquad (8.26) \\ &= \text{Present Value of project } g \text{ (times } K) \end{aligned}$$

$$\begin{aligned} \mu_k^* &= K \sum_{t=1}^{\infty} (-a_{tf})(1 + r)^{-t} \\ &= \text{Present Value of project } f \text{ (times } K) \qquad (8.27) \\ &\geq 0. \end{aligned}$$

Thus we conclude that the dependent project (about which there is a separate choice) must have a nonnegative present value.

We can make no such assertion about the independent project. All that may be said about it may be seen from a rearrangement of equation

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[19]Fractional acceptance of the independent project would be dealt with, as before, either by its exclusion or its inclusion *in toto*.

(8.26) after substituting equation (8.27) into it.

$$\mu_g^* = K[\sum_{t=1}^{\infty}(-a_{tg})(1+r)^{-t} + \sum_{t=1}^{\infty}(-a_{tf})(1+r)^{-t}] \geq 0 \quad (8.28)$$

That is, the present value of project g plus the present value of project f, computed at the rate r, adds up at least to zero.[20] This is the value of the corresponding compound project.[21]

3. Neither project is accepted: that is,

$$x_g^* = 0, \qquad x_f^* = 0.$$

Although the slack variable of restriction (8.22) now appears in the solution, constraint (8.21) is satisfied as an equation. Hence

$$\mu_g^* = 0, \qquad \mu_k^* \geq 0.$$

From constraint (8.23) we know that

$$\mu_k^* \geq \hat{a}_f - \sum_{t=1}^{T}\rho_t^* a_{tf} = K\sum_{t=1}^{\infty}(-a_{tf})(1+r)^{-t} \quad (8.29)$$

or, that μ_k^* is at least as large as the present value of project f (multiplied by the conversion factor K). Similarly, from constraint (8.24) we are able to state

$$\mu_g^* - \mu_k^* \geq \hat{a}_g - \sum_{t=1}^{T}\rho_t^* a_{tg} = K\sum_{t=1}^{\infty}(-a_{tg})(1+r)^{-t}. \quad (8.30)$$

Since $\mu_g^* = 0$ and $\mu_k^* \geq 0$, that is, $\mu_g^* - \mu_k^* = -\mu_k^* \leq 0$, we may write

$$K\sum_{t=1}^{\infty}(-a_{tg})(1+r)^{-t} \leq -\mu_k^* \leq 0 \quad (8.31)$$

which implies that the present value of independent project g is non-positive. Further, by adding relation (8.29) to (8.31) we obtain

$$K[\sum_{t=1}^{\infty}(-a_{tf})(1+r)^{-t} + \sum_{t=1}^{\infty}(-a_{rg})(1+r)^{-t}] \leq 0, \quad (8.32)$$

which implies that the present value of the sum of projects f and g must be zero or less, as we concluded before.

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[20]Since $K > 0$.

[21]Again, appearance of fractional projects may be dealt with as before except that a fractional independent project may not be eliminated if its dependent project is to be retained.

In this section we have seen some rather startling implications of our model for the selection of investment projects. Contrary to the commonly espoused form of the present value criterion, which applies only for independent investment alternatives (an assumption that is rarely stated), it is possible that the optimal program will reject some project with a positive present value or that it will accept a project with a negative present value. The former can take place when, in a mutually exclusive set of projects, some projects with positive present values are inferior to an accepted project which has a positive present value greater than any other in that set. A project with negative present value may be accepted when it makes possible another — a dependent project, in our language — whose present value is positive and large enough to make the present value of the dependent-independent project combination also positive. Since it is frequently impossible to treat all combinations of contingent projects separately, this alteration to the usual investment criteria may be of considerable significance.

8.4. The Number of Fractional Projects

For formal completeness we take up the issue of the maximum number of fractional projects which may appear in the optimal solution. The reasoning is analogous to that of Section 3.8. We limit ourselves to a consideration of independent projects since the case of contingent sets of projects, while more complex, is conceptually no different as has been shown in the theorem of Section 3.8. The importance of the subject is derived from its implications for the models of Chapter 9.

Assuming a total of n possible investment projects, there are n variables x_j and n slack variables derived from upper bound constraints on the x_j. In addition, we now have T quantities v_t, the amounts lent, and T quantities w_t, the amounts borrowed. The total number of equations is $T + n$, one for each year and one for each project, and the total number of variables is $2n + 2T$.

We have one additional consideration here not present in the Lorie and Savage problem. The v_t and w_t cannot appear simultaneously in the solution.[22] This implies that the additional values of x_j required with the appearance of fractional projects (whose slack must also be present in the solution) can come only at the expense of either the v_t or w_t.

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[22]The vectors associated with v_t and w_t are constant multiples of each other and hence are linearly dependent, preventing both from appearing together when solution procedures are restricted to the examination of basis sets of vectors only.

More precisely, the number of fractional projects is less than or equal to the number of time periods in which *neither* lending nor borrowing takes place.

The model given by equations (8.1), that is, without interrelated projects, will always yield an optimal solution without fractional projects. Should no lending or borrowing take place during some period, we may still disregard the fractional project since its contribution to the program is zero. We shall then be left with a degenerate solution. Difficulties may arise only when the fractional project is the independent member of a contingent set. While this situation calls for some adjustment of the type discussed earlier in relation to fractional projects, it is strictly a minor issue.

8.5. Choice of the Horizon

The basic horizon model of Section 8.1 requires selecting a time period prior to which outlays and revenues of potential investments are stated explicitly but beyond which the actual flows associated with these investments are collapsed into a single quantity, the horizon values. Our subsequent discussion in Section 8.2 showed that in this simple situation all flows might have been represented by their present values, and that in the absence of contingency relationships between investments there was truly no need for the use of the linear programming apparatus. The reason for this simplification is contained entirely in the assumption of perfect capital markets; that is, the situation in which lending and borrowing can take place without limit at a given interest rate. This assumption will be dropped in the models of the next chapter. Prior to embarking on these it will be instructive to mention some properties that are desired of that point in time, T, which is to serve as the horizon.

In order to unhook the infinite chain of actions and their consequences in the model of the firm's investment decisions, we seek a point in time such that the decisions which call for implementation before this date will be exactly the same, whether or not events past that moment are treated explicitly or implicitly (and hence partially ignored). More concretely, and in terms of our model, we seek a value of T such that the set of accepted projects having outlays or revenues in year T or sooner are exactly the same whether the model makes use of an infinite horizon or a horizon set at T.

In dynamic models in general such a horizon does not necessarily

exist, or there may be many of them.[23] If there are several, the earliest having this property may be designated as the preferred one because of information problems to be discussed in the next section.

As we have stated earlier, the basic model (8.1) for independent projects was not proposed for decision making since the simpler rule of choosing projects according to the sign of their present values — to which it is equivalent — is much to be preferred. Nevertheless it is useful to look at the horizon problem in its simplest setting. We notice first that when independent projects alone are to be considered, the choice of the horizon is completely arbitrary. Indeed, setting $T = 0$ has the effect of duplicating the rule of choosing all projects whose net present value at the rate r is positive and rejecting all projects whose net present value is negative.[24]

Presence of sets of mutually exclusive projects and contingent sets of projects may complicate matters. There is no real difficulty if all members of a set of mutually exclusive projects have their earliest flows in the same year. However, for example, a fossil fuel electric power plant to be constructed now and a hydro-electric or nuclear power installation at some future date may also be competing alternatives. If the horizon cuts off the earliest flow of the later project, then provision must be made to ensure consideration of both projects by the model. This may be accomplished by stating *all* horizon values of investment possibilities in the functional, even though outlays and/or revenues are stated within the restrictions of the model for only some of them. The restriction identifying the mutually exclusive set, for example relation (8.17), must, of course, be stated explicitly.

Contingent sets of projects create the same problem as do mutually exclusive projects and are handled much the same way. If, for example,

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[23]See F. Modigliani and K. J. Cohen, in M. J. Bowman (ed.), "The Significance and Uses of *Ex Ante* Data — a Summary View," *Proceedings of the Conference on Expectations, Uncertainty and Business Behavior* (New York: Social Science Research Council, 1958), pp. 151 ff.; and F. Modigliani and F. E. Hohn, "Production Planning Over Time and the Nature of the Expectations and Planning Horizon," *Econometrica*, XXIII, No. 1 (January, 1955), pp. 46–66.

[24]The model becomes:

$$\text{Maximize} \sum_j \hat{a}_j x_j$$

$$\text{Subject to} \quad 0 \leq x_j \leq 1.$$

The $\hat{a}_j$ are the present values since the horizon is the present, which also eliminates the terms v_0 and w_0 from the functional. This problem is solved by setting those x_j to one for which $\hat{a}_j$ is nonnegative, and letting all other x_j equal zero.

an investment today makes possible another tomorrow, then the horizon must either be later than both dates, or other provision for the second project must be made. Otherwise the earlier project may be rejected because it is unprofitable by itself, even though the combination of the two is eminently desirable. Using the method of "compound projects"[24] we have again only mutually exclusive alternatives, discussed above. Keeping the project identifications but stating the contingency relations, as in inequality (8.20), still permits setting an arbitrary horizon so long as the horizon value of the dependent project is included in the functional.

Unfortunately, the situation is not so simple when financial market imperfections are introduced into the model. Before turning to these, we wish next to dispose of a difficult pragmatic point relating to data sources for models of this type.

8.6. Obtaining Data for Future Investments

The model we have formulated appears to contain an element which contradicts one of the tenets of business planning cited at the outset of this chapter. Concern should be over what requires immediate implementation, and, by indirection, anything that does not call for action today should be disregarded. Specifically, implementation of the basic model of equations (8.1) calls for determination of the parameters a_{tj} and $\hat{a}_j$ for projects whose flows do not begin until some time in the future. Why should anyone estimate these quantities when action on such projects is not required in the current period? The answer here is that these quantities, insofar as they are not required, will not be determined in advance.[25] However, they may be necessary for making decisions which require implementation today,[26] and the models of imperfect capital markets of the next chapter will show these situations to be less uncommon than is generally supposed.

A general observation we may make is that when the need for some information is imperative it is somehow provided, to at least some degree of the needed accuracy. More resources may be spent on obtaining it,

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[24]See p. 11.

[25]The only projects with flows starting past the current period that need to be considered are those having some contingency relationship to current projects, as was shown above.

[26]See Gort, *op. cit.*

and one may settle for less than what is desired under "ideal" circumstances.[27]

In the interviews conducted by the author with individuals of large corporations having responsibilities in this area, the point was elicited that data of this type are recognized to be essential for forward planning, and hence estimates are supplied to fulfill these planning needs.

In the first instance, those projects which are the primary concern of long-range planners have long lead-times due to the development work required. This means that estimates are available many years in advance as to what products and processes will be "coming along." Although no precise cost and revenue figures can be made very far in advance, preliminary estimates are routinely made, and their implications for planning are analyzed.

For the very distant future one of the interviewed companies makes periodic projections of the investment required for capital maintenance and for maintenance of the current or projected rate of capacity expansion. These estimates are made to quantify the corporation's goals and objectives and are tempered by separate projections of demographic and economic variables. Detailed estimates are not made. Instead, totals are estimated by broad functional divisions, and certain balance relations are superimposed upon them. The purpose of these estimates is to facilitate "reservation of blocks of funds" for unplanned investments for the integration of financial and physical plans, and to make company officials aware of the probable existence of capital projects in the future, which will have to be considered along with those already identified.

It must be emphasized, once again, that the problem concerning data pertaining to investments in the more distant future does not arise under the assumptions of this chapter except as they involve contingent chains of projects stretching into the future. However, they do arise in conjunction with the models of the next chapter which consider various forms of capital rationing.

One final comment relating to an alternative strategy regarding the estimation of future cash flows may be offered. A reversal of the direct approach suggested above may be employed to establish the relevance of such data. This may be accomplished by use of parametric programming, as outlined in Chapter 7, to obtain an indication of the

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[27]The assumptions of economic theory usually include "perfect knowledge." See also C. C. Holt, "Forecasting Requirements from the Business Standpoint," *ONR Research Memorandum*, No. 56 (Pittsburgh: Graduate School of Industrial Administration, Carnegie Institute of Technology, October, 1957).

periods of time and related circumstances[28] when constraints relating to capital or other resources are likely to be binding. When these conditions have been identified it would appear useful to intensify planning activities with respect to those periods in order to make possible a comparison between the relevant alternatives.

8.8. Conclusion

This chapter has been devoted to a reformulation of the capital budgeting problem by casting it into an explicitly intertemporal mode. In this reformulation we have also been able to accomplish some breaking down of the inputs into the Lorie and Savage model into more basic quantities. We have seen that this new formulation carries with it only expositional value when the assumptions of perfect capital markets and independent investments are made. The value of the approach is partially revealed when nonindependent investments are also included. In this simple, yet commonplace, circumstance, the traditional investment criteria of present values and internal rates of return become extremely cumbersome, if not useless, after the required modifications. In the next chapter, in which we take up capital market imperfections and their effect on the optimal choice of investments, we shall see further the reasons for the more fundamental approach to investment problems which programming methods make possible.

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[28]For example, of manpower limitations.

CHAPTER 9

Application of Programming Methods to Capital Budgeting Under Imperfect Capital Markets[1]

In a world of perfect capital markets, as we have seen in the preceding chapter, the investment decision can be completely divorced from the financing problem. Indeed, the latter does not exist, for it is supposed that funds come out of a limitless tap under constant pressure (that is, the interest rate). When the assumption of perfect capital markets is relaxed, or even when the presence of fixed costs such as the organizational costs of negotiating bank loans or the fixed charges associated with bond flotations are considered, the interrelationships between the finan-

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[1]The term "imperfect capital markets" is to be taken seriously since in a world of perfect capital markets, as Modigliani and Miller have suggested [see "The Cost of Capital, Corporation Finance and the Theory of Investment," *American Economic Review*, XLVIII, No. 3 (June, 1958), pp. 261–97], there is no optimal capital structure. The empirical issue of whether this is the case or not is not our concern here. Rather, this chapter is an attempt to apply the apparatus developed earlier from a managerial point of view.

There is, of course, one further difficulty with any optimization model for a situation involving imperfect capital markets. This is the question of what is the appropriate function to be maximized. In the case of perfect capital markets the

cial decisions and decisions with respect to physical investment cannot be ignored.[2]

It is the purpose of this chapter to analyze the capital budgeting problem under various forms of capital rationing and capital market imperfection. Hart, who asserts that "capital-rationing is an essential trait of the capitalist economy,"[3] also states that this phenomenon is clearly an effect of uncertainty.[4] When movement of borrowed capital is inhibited by other than profitability considerations, the reason is usually that the supplier of funds regards the safety of his loan to be less than complete, even though the borrower may regard his prospects as certain.

> The range of choice open to the management of a business with respect to the form of its financing is limited and circumscribed by the decisions of suppliers of funds. These decisions, in turn, appear to be based upon the characteristics of the business as they are seen by suppliers, and upon the generally accepted financial standards that are applied to the business by suppliers of funds. Within this range, the management of an enterprise exercises its discretion in choosing from among those forms of financing that are available to it in the market.[5]

In undertaking this analysis of the implications of imperfect capital markets we shall not go so far as to treat the outcomes of investments as

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issue does not arise [see, for example, J. Hirshleifer, "On the Theory of Optimal Investment," *Journal of Political Economy*, LXVI, No. 4 (August, 1958), pp. 329–52], since maximizing the present or terminal value of the firm can be shown to be equivalent to maximizing the economic welfare of the owners regardless of differences in their time preferences. Under imperfect capital markets these cases are not necessarily equivalent and may not even be defined.

Without attempting to solve the problem of the appropriate criterion we shall here simply assume that the criterion is given from outside, and, for convenience, as well as to preserve continuity with our earlier formulation, we shall continue to maximize the terminal value. Clearly, alternative formulations could be used.

[2]Many writers make this point. See, for example, Neil H. Jacoby and J. Fred Weston, "Factors Influencing Managerial Decisions in Determining Forms of Business Financing: An Exploratory Study," *Conference on Research in Business Finance* (New York: National Bureau of Economic Research, Inc., 1952), pp. 145–46.

"The two types of decisions [that is, the determination of the amount to invest and the forms of financing] clearly are interdependent. The particular forms and variants of financing that are available at any time to business concerns have an influence of considerable importance upon the amount of the current demand for funds."

[3]*Op. cit.*, p. 33.

[4]*Ibid.*, p. 40.

[5]Jacoby and Weston, *op. cit.*, p. 166.

stochastic variables.[6] Our more limited approach to uncertainty is through a consideration of the attitudes of suppliers of capital toward risk. Although explicit treatment of the full effects of uncertainty is surely desirable, the solution of the simple problem is a necessary first step.

Specifically, we shall again be concerned with investments having possibly irregular patterns of flows of costs and revenues, and with the integration of the financing decisions with the decision to make physical investments. Scant attention has been paid in the literature to the multi-period investment problem under less than perfect capital markets and still less in a form in which it may ultimately become a managerial tool.[7] It is our intention to maintain a reasonably operational framework, although no pretense is intended that these methods may be put to use without further refinement.

In the discussions which follow, the problems created by complementarities and competitiveness between projects are ignored. The method of analysis of the preceding chapter may be applied to these problems also in the models which follow, and little would be gained by a repetition of this subject.

9.1. Absolute Limits on Borrowing

The simplest way in which the assumption of perfect capital markets may be dropped — although it may also be the severest — is the imposition of absolute limits on the amount of debt the firm may owe at any time. It is the simplest way in a logical sense, and, as we shall see, its structure is of a form which may easily be extended to situations of rising supply curves for funds. It may be more rigid than is generally found in the real world, although it is by no means without basis in actuality.

Charnes, Cooper and Miller, in their treatment of this subject,[8] imposed a liquidity constraint on the financial management of the firm in the form of a minimum cash balance. The rationale for such a hedge

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[6]It is by no means obvious that the use of sensitivity analysis and generalized parametric methods (which permit use of nonlinear functions and also apply to modifications of interior coefficients) is more limited than the assumption of known probability distributions.

[7]An exception again is the article by Charnes, Cooper and Miller, "Application of Linear Programming," *op. cit.*

[8]*Op. cit.*, p. 25 ff.

against chance events may be attributed to the typical financial ratio tests applied by bankers and investment analysts. Another form of limit on borrowing they suggested was in terms of a maximum amount of trade credit suppliers are willing to extend. Counterparts to these borrowing limitations in capital planning are not lacking.[9]

Perhaps a more common form of limitation on borrowing by firms is that self-imposed regulation under which capital expenditures may be made only from retained earnings and other internal sources. Heller's study may again be cited:

> Most firms are not in this happy position [of having sufficient liquidity or timely profits, access to outside financing, or leeway in dividend policy to increase investment in response to attractive opportunities]. For them, . . . the amount set aside for plant and equipment spending is in the nature of a residual, especially where they cannot or will not resort to outside financing.[10]

Limiting expenditures to internally generated funds is designed to avoid dilution and possible loss of control by management or the present ownership which may accompany outside financing. Externally imposed borrowing limits, aside from those associated with mortgages and other collateral,[11] sometimes apply to public utilities for somewhat different reasons. The Public Utilities Holding Act of 1935 empowers the Securities and Exchange Commission to prescribe capital ratios for corporations subject to its jurisdiction under the act. While the number of such companies has dwindled almost to the vanishing point, state regulatory bodies have assumed comparable powers. With these points as justification, we turn to the model itself.

We may state borrowing limits as an additional set of constraints to the basic horizon model given in equations (8.1) and repeated below. Let the earlier definitions of the variables be retained and introduce the symbol B_t to denote the maximum amount of debt permitted in period t.

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[9]As one example which approximates the situation of limited credit in the supply of capital equipment one may cite the recent agreement between the Flying Tiger Line, an air cargo carrier, and the Canadair Ltd. subsidiary of General Dynamics Corporation. The manufacturer of the CL-44 aircraft in effect loaned the operator sufficient funds for a specific number of planes, although in this instance the transaction was complicated by the addition of a feature to permit conversion of the equipment bonds into common stock. Other aircraft purchase agreements might also be cited to exemplify this type of limitation on borrowing.

[10]*Op. cit.*, p. 98.

[11]These were discussed in Chapter 6.

A model that can then be synthesized to study the topics at issue is

$$\begin{aligned}
&\text{Maximize} && \sum_j \hat{a}_j x_j + v_T - w_T \\
&\text{Subject to (a)} && \sum_j a_{1j} x_j + v_1 - w_1 \leq D_1 \\
&\text{(b)} && \sum_j a_{tj} x_j - (1+r) v_{t-1} + v_t \\
& && \qquad + (1+r) w_{t-1} - w_t \leq D_t, \\
& && \qquad\qquad t = 2, \ldots, T \qquad (9.1) \\
&\text{(c)} && 0 \leq x_j \leq 1, \quad j = 1, \ldots, n \\
&\text{(d)} && w_t \leq B_t, \quad t = 1, \ldots, T \\
&\text{(e)} && v_t, w_t \geq 0, \quad t = 1, \ldots, T.
\end{aligned}$$

We have utilized this formulation in order to effect easy comparisons with earlier models like (8.1). Thus, note that models (8.1) and (9.1) are identical except for new constraints (9.1d). The immediate effect of the added constraints, if they are operative, is to increase the effective rates of compound interest at which the investment projects are implicitly evaluated. The dual to this problem will reveal this when we contrast it with earlier models like (8.2).

Denoting the dual variables that are associated with these new constraints by β_t, the system (8.2) becomes

$$\begin{aligned}
&\text{Minimize} && \sum_t \rho_t D_t + \sum_j \mu_j + \sum_t \beta_t B_t \\
&\text{Subject to (a)} && \sum_t \rho_t a_{tj} + \mu_j \geq \hat{a}_j, \qquad j = 1, \ldots, n \\
&\text{(b)} && \rho_T \geq 1 \qquad (9.2) \\
&\text{(c)} && -\rho_T + \beta_T \geq -1 \qquad [\text{that is, } \rho_T - \beta_T \leq 1] \\
&\text{(d)} && \rho_t - (1+r)\rho_{t+1} \geq 0, \qquad t = 1, \ldots, T-1 \\
&\text{(e)} && -\rho_t + (1+r)\rho_{t+1} + \beta_t \geq 0, \qquad t = 1, \ldots, T-1 \\
&\text{(f)} && \rho_t, \beta_t, \mu_j \geq 0.
\end{aligned}$$

If in year t we have $w_t^* < B_t$ so that borrowing is less than the maximum amount permitted by the debt limit, B_t, applicable in that period then the optimal value of the associated dual variable is zero; that is, $\beta_t^* = 0$. However, if borrowing takes place up to the limit B_t, then $\beta_t^* \geq 0$. Note, in particular, that condition (8.2c) gives way to (9.2c)

so that values $\rho_T^* > 1$ are now possible while, via (9.2b), values of $\rho_T^* < 1$ still cannot occur. This is, of course, a reflection of the fact that a debt limitation in amount B_T can be applied to the horizon date in such a way that it effectively removes otherwise profitable opportunities — perhaps in earlier periods — and thereby raises the opportunity cost of funds as implicitly evaluated by the figure secured for ρ_T^*.

We can make the point somewhat clearer by observing, as before, that the ratio

$$\frac{\rho_t^*}{\rho_{t+1}^*} \geq 1 + r$$

can be used to determine an "annual" interest rate for every period $t \leq T - 1$ and by relating it to the borrowing and lending rates, r, that are applicable up to the debt limitations B_t. These ρ^* values implicitly determine the rates that are applicable to the actual flows a_{tj} for each project j in year t and subsequent years in the discounting procedure contained in the dual problem.

More explicitly, we may develop the ρ_t^* recursively from restrictions (9.2d and e), which we now display separately via the expressions

$$\rho_t^* \geq (1 + r)\rho_{t+1}^* \tag{9.3}$$

and

$$\rho_t^* \leq (1 + r)\rho_{t+1}^* + \beta_t^* \tag{9.4}$$

and observe that all variables in these expressions necessarily assume positive values (since $r \geq 0$) with the possible exception of β_t^*. Of course, $\beta_t^* \geq 0$ necessarily holds, via the conditions (9.2f). If $\beta_t^* = 0$ for all t, then, of course, we are back to equations (8.7). To allow for the possibility of some $\beta_t^* > 0$, however, we must then replace the simpler equations (8.7) by

$$\rho_t^* = (1 + r)^{T-t} + \sum_{\tau=t}^{T} (1 + r)^{\tau-t}\beta_\tau^* \tag{9.5}$$

in order that the ρ_t^* may properly reflect the opportunity costs associated with borrowing up to a ceiling, B_t (as necessarily occurs when $\beta_t^* > 0$).

The significance of expression (9.5) may be brought out in yet another way by observing that the ρ_t^*, aside from departing from compound interest at the annual rate, r, involve terms from years other than the year, t, with which they are associated. The opportunity cost of funds of a particular year may rise, as we have just seen, above the market

rate of interest. Our earlier analysis of the Lorie and Savage models has shown that present values, computed with assumed rates of interest, are insufficient for investment decisions in the presence of fixed budgets. Here, in contrast, a present value criterion would work if the rates were known. But here we must depart from the classical assumptions of investment analysis and regard these values, rather, as unknowns to be determined as part of the problem being solved.[12]

The first observation we may make is that flows a_{tj} are now generally discounted at different interest rates for different years, depending on the lending or borrowing activities[13] and on the investment opportunities that must be rejected because of the limit on the amount of borrowing. We see this by using equation (9.5) to express the value of an accepted project as follows:

$$\mu_j^* = \hat{a}_j - \sum_{t=1}^{T} \rho_t^* a_{tj}$$

as before; hence

$$\mu_j^* = \hat{a}_j + \sum_{t=1}^{T} (-a_{tj})[(1+r)^{T-t} + \sum_{\tau=t}^{T} (1+r)^{\tau-t}\beta_\tau^*]. \qquad (9.6)$$

Accepting, for the moment, the validity of expression (8.10) for the horizon value of flows subsequent to the horizon — that is, if we accept

$$\hat{a}_j = \sum_{t=T+1}^{\infty} (-a_{tj})(1+r)^{T-t},$$

then we may rewrite the value of an accepted project as

$$\mu_j^* = \sum_{t=1}^{\infty} (-a_{tj})(1+r)^{T-t} + \sum_{t=1}^{T} \beta_t^* \sum_{\tau=1}^{t} (-a_{\tau j})(1+r)^{t-\tau}, \qquad (9.7)$$

which is the horizon value of the project computed at the market rate of interest, r, plus an additional term which we shall explain presently.

For rejected projects we necessarily have $\mu_j^* = 0$, while for accepted projects we can only assert that $\mu_j^* \geq 0$. In contrast, an accepted project has $x_j^* > 0$ and so (9.7) is satisfied. But we can then see from (9.7)

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[12]On this point, see also Charnes, Cooper and Miller, "Application of Linear Programming," *op. cit.*, n. 49, p. 34.

[13]This gives rise to the controversy over whether the lending or the borrowing rate of interest is the more appropriate measure of the opportunity cost of funds. On this subject, see Lutz and Lutz, *op. cit.;* Harry Roberts, "Current Problems in the Economics of Capital Budgeting," *Journal of Business*, XXX (January, 1957), pp. 12–16; and J. Hirshleifer, "On the Theory of Optimal Investment," *op. cit.*

that the flows in individual years for a given accepted proposal can be accorded an interpretation in terms of a two-part evaluation. In the first, their contribution to the present or horizon value is measured at the market rate of interest, just as in the classical theory of investment. But the second collection of terms in (9.7) causes a departure from these values if any $\beta_t^* > 0$. The values of μ_j^* increased thereby, which are associated with the corresponding x_j^*, must then reflect what we shall refer to as the premium, β_t^*, allowing for suitable compounding, which applies to the related borrowing ceiling.

For simplicity in notation, we define an additional variable A_t:

$$A_t = \sum_{\tau=1}^{t} (-a_{\tau j})(1 + r)^{t-\tau}. \tag{9.8}$$

Expressed in words, A_t is the compound value, as of the year t, of all flows associated with project j prior to the close of the current year. Using this variable, the second term of equation (9.7) may be rewritten as

$$\sum_{t=1}^{T} \beta_t^* A_t. \tag{9.9}$$

We know that $\beta_t^* \geq 0$, with $\beta_t^* > 0$ implying that $w_t^* = B_t$ so that borrowing takes place up to the limit B_t. Moreover, when $\beta_t^* > 0$, this premium is applied to project j whether or not it has any flows in year t itself. For example, a project may become more valuable because it generates revenues in earlier years and hence reduces the pressure of the borrowing constraint in the later year. Alternatively, the value of a project is lowered because its costs in earlier years are a drain on resources which could, had they been preserved, have lessened the pinch on funds in the given year. All this, of course, is dependent on the possibility of lending, that is, transferring funds to later years.

We now observe an important point in connection with equation (9.7) which may help to clarify some of the controversies that have occurred in the literature dealing with capital budgeting. The point is that a project need not be rejected merely because it has a negative present value (or horizon value),

$$\sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{T-t} < 0, \tag{9.10}$$

as determined by the *market* interest rate, r, only. The criteria of acceptance must also include the premium, β_t^*, so that if

$$\sum_{t=1}^{T} \beta_t^* A_t > 0 \tag{9.11}$$

then we can still satisfy $\mu_j^* \geq 0$ in (9.7) even when we admit

$$\sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{T-t} < 0,$$

provided only that we continue to satisfy

$$\mu_j^* = \sum_{t=1}^{T} \beta_t^* A_t + \sum_{t=1}^{\infty} (-a_{tj})(1 + r)^{T-t} \geq 0. \tag{9.12}$$

Of course, if all $\beta_t^* = 0$, this would be reduced to (8.11), and acceptance of a project with a negative present value could not occur. When positive premiums β_t^* occur, however, the relevant condition is

$$\sum_{t=1}^{T} \beta_t^* A_t \geq \sum_{t=1}^{\infty} a_{tj}(1 + r)^{T-t} \tag{9.13}$$

and, then — subject to this condition — the reason for possible acceptance of a project with net negative present values at market rate, r, becomes clear. It may generate inflows (that is, revenues) at times when these are most needed. Hence such an alternative can become attractive even though it would never be acceptable in the absence of borrowing limits.[14] The converse is also likely to occur. A project may be rejected even though its present value at the market rate of interest is positive.[15]

The validity of equation (8.6), as mentioned earlier, hinges on the legitimacy of using the assumed rate of interest, r, in the computation of the horizon values of the flows of the projects subsequent to the horizon, that is, in the substitution of

$$\hat{a}_j = \sum_{t=T+1}^{\infty} (-a_{tj})(1 + r)^{T-t}$$

into the expression for μ_j. While the practical significance of this issue may be negligible (especially with sufficiently large T) its theoretical implications are derived from the nature of the chosen horizon and are

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[14]Or other financial market imperfections.

[15]As was pointed out in connection with the Lorie-Savage models, the common measures for ranking investment alternatives may be misleading in the presence of capital rationing. Aside from familiar criticisms leveled at the internal rate of return as a ranking device (cf. Hirschleifer, *op. cit.*, p. 340), present values are equally deficient here. As pointed out above, either they lead to wrong decisions, given an assumed rate of interest for discounting, or the rates at which the discounting is to be done must be solved for, much as they are here. Numerical examples of these conclusions are presented in the appendix to this chapter.

related further to a subject we have not discussed so far, namely, the dividend policy of the firm. We touch on both topics in the next section.

9.2. Choice of the Horizon and the Implications of Dividends

In an earlier model analyzed in Chapter 8 in which unlimited lending and borrowing opportunities were assumed, the truncation of the problem by means of an imposed horizon presented no real difficulties. This fortunate circumstance no longer applies here. Although our criticism of the common economic evaluators of capital investments still stand, conditions must be laid down to assure that our own procedure leads to correct decisions.

The rate taken to be appropriate in computing the horizon values $\hat{a}_j$, above, is the lending-borrowing rate used in the models of the preceding and present chapters. However, this rate is not the proper one if there are effective limits on borrowing. Stated differently, the horizon must not be chosen in such a way as to by-pass the borrowing limits.[16] The problem, incidentally, is not the degree of inaccuracy, if any, in the μ^* but rather the incorrect investment decisions which might be made because binding constraints are neglected.

By their very nature, limits on borrowing (or lending), like spending ceilings in the case of the Lorie and Savage models, interrelate all investment decisions for the indefinite future. Instead of attempting to formulate a method of finding horizons which serve "sufficiently well"[17] we shall argue that the effectiveness of limits on borrowing in distant time periods is largely the result of the dividend policies pursued.[18]

The separability of the investment decision from that of consumption

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[16]It may be possible to lay down conditions under which the technique of locating subhorizons suggested in Charnes and Cooper, *Management Models and Industrial Applications of Linear Programming, op. cit.*, Chapter XI, may be utilized. These methods yield horizons which have the properties discussed in Section 8.5 above.

[17]For example, horizons such that the borrowing constraints are "epsilon irrelevant" after year T, in the language of F. Modigliani and K. J. Cohen, "The Role of Anticipations and Plans in Economic Behavior and Their Use in Economic Analysis and Forecasting," *op. cit.*

[18]This is not to be interpreted as an exact statement. For example, it excludes firms not generating any cash throw-off and not paying dividends. However, in a later model (Section 9.5, below) we shall approximate uncertainty more closely by substituting equity financing for long-term borrowing, and by limiting short-term borrowing to the short term. This makes the effect of a possible short-term borrowing limit beyond the horizon a less important problem.

under the assumptions of the basic horizon model of Section 8.1 — that is, under the assumption of perfect capital markets — is fundamental here. It can be shown that under perfect capital markets, rational behavior and perfect certainty, dividend policy will have no effect on the total present value of the enterprise, given the investment policy of the firm.[19] Thus, an investor can adjust to his preferred pattern of consumption over time by purely financial transactions at the market rate of interest. Alternatively, the firm can adjust, without cost, to a pattern of dividends by means of its own financial dealings and at the same rate of interest. This no longer holds when borrowing limits are imposed.

Generally, in situations of less-than-perfect capital markets, the preferred consumption pattern (indifference map) of the owners — together with the set of investment opportunities — will determine the optimal investment decision.[20] In contrast, as a practical matter, when there are many owners and also many firms, borrowing limits may become ineffectual if owners are willing to "lend" to the firm in the form of foregone dividends. While this may not be attractive to all owners, a suitable matching of owners and firms may come about by means of trading in the shares of such companies so that the timing of dividends is optimal both to the firms and to their owners. This would suggest that capital rationing of this form is important only in the near term, during which adjustment may be incomplete.[21] In any case, we turn our attention now to a somewhat more meaningful way of capturing some of the effects of uncertainty as they relate to suppliers of funds in the form of a rising supply schedule for borrowed funds.

9.3. Sloping Supply Schedule for Funds

A more realistic approximation to uncertainty than absolute limits on borrowing is a rising supply curve for funds.[22] Here the higher rates

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[19]See M. H. Miller and F. Modigliani, "Dividend Policy, Growth, and the Valuation of Shares," *Journal of Business*, XXXIV, No. 4 (October, 1961), pp. 411–33.

[20]See, for example, Hirshleifer, *op. cit.*

[21]Hirshleifer regards this case "empirically significant only in the short run." *Op. cit.*, p. 339.

[22]"If investors providing capital for a firm normally based their decisions on study of the would-be borrower's character and prospects, we might expect to find a smooth rise in interest rates as the borrower's debt rose in proportion to his own equity. This expectation arises from the basic assumption that the investor is uncertain about the businessman's intention and ability to repay." A. G. Hart, *op. cit.*, p. 43.

of interest associated with larger amounts borrowed are interpreted to be a form of risk premium, compensation for the increased risk of default.[23]

The supply schedule may be built into the model by a series of restrictions of the form of borrowing limits, such as equations (9.1d), and by the addition of terms corresponding to borrowing at different rates of interest. The form of the actual restriction will be a step-function, although, as will be apparent, this is equivalent to a piece-wise linear supply schedule for funds.

Let r_i be the interest rate which applies to the ith step of the (marginal) supply curve,[24] and let w_{it} be the amount borrowed in this step in year t. Further, let B_{it} define the upper limit of the ith step. The model for selecting investment projects then becomes

$$\text{Maximize} \quad \sum_{j=1}^{n} \hat{a}_j x_j + v_T - \sum_{i=1}^{m} w_{iT}$$

$$\text{Subject to (a)} \quad \sum_{j=1}^{n} a_{1j} x_j + v_1 - \sum_{i=1}^{m} w_{i1} \leq D_1$$

$$\text{(b)} \quad \sum_{j} a_{tj} x_j - (1 + r) v_{t-1} + v_t + \sum_{i} (1 + r_i) w_{i,t-1} - \sum_{i} w_{it} \leq D_t, \qquad (9.14)$$

$$t = 2, \ldots, T$$

$$\text{(c)} \quad w_{it} \leq B_{it}, \qquad t = 1, \ldots, T; \qquad i = 1, \ldots, m$$

$$\text{(d)} \quad 0 \leq x_j \leq 1, \qquad j = 1, \ldots, n$$

$$\text{(e)} \quad v_t, w_{it} \geq 0$$

where $\hat{a}_j$ is computed at a rate r to be discussed.

The rates of interest of the steps of the marginal cost of funds schedule are ordered so that $r_{i-1} < r_i < r_{i+1}$. This relation between the rates of interest on adjacent steps eliminates the necessity for stating lower limits on the amount of borrowing for each of the steps. The alternative to borrowing at a rate r_i is borrowing at a lower, and

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[23] We are assuming that stockholders cannot borrow at rates less than or equal to the rates which the firm must pay. See Modigliani and Miller, "The Cost of Capital, Corporation Finance, and the Theory of Investment," *op. cit.*

Our reason for discussing absolute borrowing limits before considering a rising supply schedule was based purely on the technical nature of the changes which must be made in the basic model to incorporate this modification.

[24] Again, these interest rates may also have time subscripts.

hence preferred rate, say r_h, with $h < i$. If any borrowing at rate r_i takes place it must be because the limited amount on each of the prior steps has been exhausted.[25]

We may show that the step function as presented is the marginal cost of borrowed funds curve implied by (or which implies) the average cost curve (usually called the supply curve) by use of a numerical example. Suppose that borrowing may take place in any amount up to $200 at a rate of interest of 10 per cent; that additional amounts, up to an extra $200, may be borrowed at 12 per cent; and finally that additional amounts, up to another $200, may be borrowed at a rate of 17 per cent. With marginal interest rates of 10 per cent, 12 per cent, and 17 per cent, the corresponding average rates in these intervals are 10 per cent, 10–11 per cent, and 11–13 per cent, as shown in Fig. 9.1

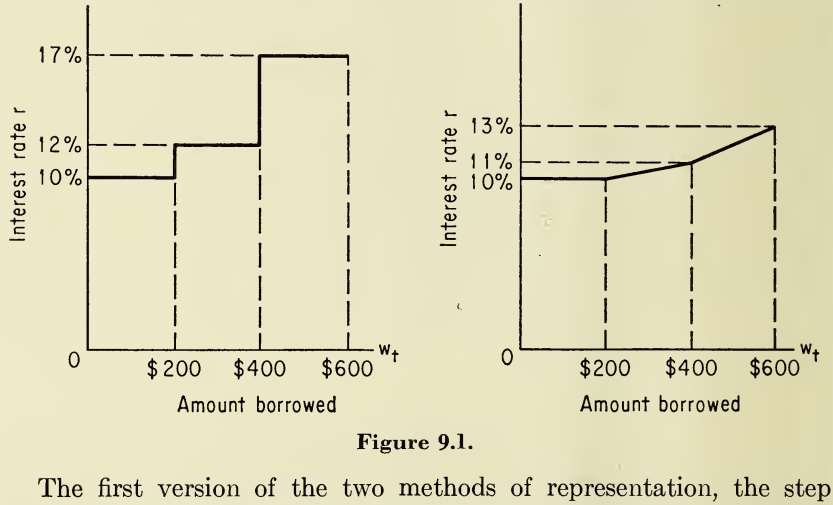

Figure 9.1.

The first version of the two methods of representation, the step function, is probably the more realistic since the different rates are more likely to derive from different sources of credit pyramided on top of of one another.[26] Theoretically, however, there is no reason why any

[25]Unless the lowest borrowing rate is at least as high as the lending rate, the model may make the firm into a financial intermediary which borrows at one rate and makes a profit by lending at a higher rate.

[26]". . . a smooth rise of interest rates with a thinning of the equity is unlikely. Instead, the firm is confronted with a series of opportunities for raising funds, each with its own interest rate and its own rules, and each — ordinarily — having fairly definite quantitative limits. That is, the entrepreneur obtains capital on a 'stepped' market" (Hart, *op. cit.*, p. 44).

shape of supply curve cannot be approximated by a sufficient number of linear segments.[27]

The dual problem again reveals the nature of the discounting process which is implicitly applied to the projects. Equation (8.8) is still valid here; that is,

$$\mu_j^* = \hat{a}_j + \sum_t \rho_t^*(-a_{tj}),$$

although the ρ_t^* now differ slightly in interpretation. They will be seen to be the *marginal* rates of interest for period t, compounded to the horizon. This may be shown if we make an assumption, necessary only for mathematical rigor, that the last step on the function for borrowed funds is "open" or, alternatively, that the limit on the step with the highest rate is never reached. Since it is always possible to extend the function beyond any rate of interest that will be utilized, nothing is lost in making this assumption for each period. In place of equations (9.2e) we will have a series of equations in the dual problem. Denoting by β_{it} the dual variable associated with the upper bound constraint on the ith step in year t, we can write the dual constraints as

$$-\rho_t + (1 + r_i)\rho_{t+1} + \beta_{it} \geq 0, \quad t = 1, \ldots, T - 1 \qquad (9.15)$$

which applies to all steps i with the exception of the last, for which there is no upper bound.

We focus our attention on the step function of the primal problem, and in particular on the last segment for a given year t for which $w_{it}^* \geq 0$. We denote the interest rate of this step by $r_i(t)$ and consider, in turn, the three possibilities. First, $0 < w_{it}^* < B_{it}$; second, $w_{it}^* = B_{it}$; and third, $w_{it}^* = 0$. In the first instance, by reason of "complementary slackness," we have both (9.15) satisfied as an equation, and $\beta_{it}^* = 0$. This gives the desired result:

$$\frac{\rho_t^*}{\rho_{t+1}^*} = 1 + r_i(t). \qquad (9.16)$$

When borrowing in any year takes place up to the limit of some step but no amount is indicated for the next higher rate of interest, then we are only able to bracket the marginal rate, as is usually the case when the average curve has a discontinuous derivative at some point. In this

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[27]See A. Charnes and W. W. Cooper, "Nonlinear Power of Adjacent Extreme Point Methods in Linear Programming," *Econometrica*, XXV, No. 1 (January, 1957), pp. 132–53. We ignore a backward rising supply curve here, although integer methods may also be used to handle these.

situation we have condition (9.15) satisfied as an equation and β_{it}^* is nonnegative. Further, from the constraint on the following step, which is satisfied as a strict inequality,[28] we have $\beta_{i+1,t}^* = 0$ and the corresponding form of constraint (9.15) is satisfied as a strict inequality. These may be stated as two conditions which may then be combined:

$$\frac{\rho_t^* - \beta_{it}^*}{\rho_{t+1}^*} = 1 + r_i(t) \tag{9.17}$$

or, since $\beta_{it}^* \geq 0$,

$$1 + r_i(t) \leq \frac{\rho_t^*}{\rho_{t+1}^*};$$

and

$$\frac{\rho_t^*}{\rho_{t+1}^*} \leq 1 + r_{i+1}(t), \tag{9.18}$$

that is,

$$1 + r_i(t) \leq \frac{\rho_t^*}{\rho_{t+1}^*} \leq 1 + r_{i+1}(t). \tag{9.19}$$

Finally, when no borrowing takes place but lending does, $\rho_t^*/\rho_{t+1}^* = 1 + r$, as before. Numerical examples of this model of a rising supply curve for borrowed funds with numerical values for the optimal dual variables are provided in the appendix to this chapter. We turn our attention next to further relaxing the assumption of perfect capital markets.

9.4. Approximation to Equity Financing

In this and the following sections we present two further efforts in the direction of increasing realism. In the first of these we consider the possibility of short-term borrowing by the firm at an interest rate *below* the rate on long-term loans which are taken as approximations to equity financing. This situation becomes nontrivial in the present context only when short-term loans are not renewable, although further short-term credit may be obtained after an interval. This might be said to correspond, roughly, to the practice by banks of requiring users of lines of credit to be fully paid up for a certain length of time at given intervals, so that the banks are not, in fact, providing the "permanent" capital of the firm.

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[28] Since $w_{i+1,t}^* = 0$.

To solve this problem of two rates of interest involves use of mixed integer programming because it requires the control not only of the level of some of the activities, but also of their *number*. The model itself is largely of the form of earlier models of this chapter. The addition of new activities of long-term borrowing to those of short-term borrowing are the chief modifications. We consider long-term borrowing, denoted by the symbol $\hat{w}_t$, to be repaid at the horizon, while short-term borrowing continues to be for one year only.[29] Aside from the special restrictions related to the integer variables which will be stated separately, the model becomes

$$\begin{aligned}
\text{Maximize} \quad & \sum_j \hat{a}_j x_j + v_T - w_T - \sum_{t=1}^{T} (1 + r_e)^{T-t} \hat{w}_t \\
\text{Subject to (a)} \quad & \sum_j a_{1j} x_j + v_1 - w_1 - \hat{w}_1 \leq D_1 \\
\text{(b)} \quad & \sum_j a_{tj} x_j - (1 + r) v_{t-1} + v_t \\
& \qquad + (1 + r_s) w_{t-1} - w_t - \hat{w}_t \leq D_t, \\
& \qquad\qquad t = 2, \ldots, T \\
\text{(c)} \quad & 0 \leq x_j \leq 1 \\
\text{(d)} \quad & v_t, w_t, \hat{w}_t \geq 0,
\end{aligned} \tag{9.20}$$

where the short-term rate of interest is r_s, and the annual long-term rate is r_e. The lending rate continues to be denoted by r. We now require an estimated quantity, B, which is as large or larger than any amount of short-term borrowing which may materialize. This value need not be known with precision. We also define artificial variables W_t by the following restrictions:

$$\frac{w_t}{B} \leq W_t, \qquad t = 1, \ldots, T. \tag{9.21}$$

These are to be added to those already set forth in (9.20) along with further restrictions that

$$W_t \leq 1, \quad \text{and integer.} \tag{9.22}$$

Finally, we require that

$$W_t + W_{t+1} \leq 1, \qquad t = 1, \ldots, T - 1. \tag{9.23}$$

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[29] As in the earlier models, loans of more than one period duration could have been incorporated into the model. Unless these offer lower annual rates, they need not be utilized with the assumed unlimited renewability of the one-year loans.

The effect of these added constraints is as follows. Since B is as large or larger than any amount conceivably borrowed in one year, if $w_t^* > 0$, then also, via (9.21), $W_t^* > 0$. But since (9.22) requires that W_t^* also be integer we can only satisfy $W_t^* > 0$ with $W_t^* = 1$. However, when $w_t^* = 0$, that is, when no short-term borrowing takes place in year t, then W_t^* may be strictly less than 1. Restrictions (9.23) then limit the one-year loans to one every two years, although different restrictions of this form could be formulated to impose a different pattern of short-term borrowing. The only possibility for continuous loans would then be by means of the long-term loans, akin to stock financing.[30]

9.5. Optimizing the Amount and Timing of Equity Financing

We finally turn to a linear programming model which treats a similar problem by means of parametric methods. For this we shall construct a model that is designed to complete the integration of the investment decision with the financing decisions by optimizing the amount and timing of stock to be issued in all years to the horizon while simultaneously choosing the optimal set of investments and the optimal amounts of borrowing and lending. For the reasons argued earlier, we omit consideration of dividends, that is, we consider a firm which pays none.[31]

Within the context of our model we shall regard stocks essentially to be loans to the firm which require no repayment. From the point of view of the firm, regarded as a separate entity, "free" funds are always welcome. However, the current owners of the firm regard additional stock desirable only if the earnings generated by the proceeds from the stock issue more than compensate the current owners for the reduction in the proportion of the total earnings which accrues to them. Translating these flows into stocks, we may say that, from the standpoint of the present ownership of the firm, additional equity financing is desirable

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[30]Since this is an integer programming model, the x_j can also be required to be integer.

[31]It would be a simple matter, from a technical point of view, to add a dividend policy of any specified dividend in each year or to make dividends a function of other variables of the model. However, especially in the latter case, to do so would make it impossible to make comparisons between several different solutions since the dividends would be evaluated differently from the terminal value of assets by owners with a different set of time-preferences. On the other hand, disregarding dividends should not be taken to imply that a firm can exist with a policy of *never* paying dividends.

only if the per-share terminal value of the firm is increased thereby; otherwise not.[32]

The per-share value of stocks will be determined by looking at the optimum horizon value of the firm without additional stock.[33] That is, we look at the value of

$$\text{Maximum } [\sum_j \hat{a}_j x_j + v_T - w_T + \hat{D}] \tag{9.24}$$

where

$$\hat{D} = \sum_{t=T+1}^{\infty} D_t(1 + r)^{T-t}.$$

The last term, $\hat{D}$, is the horizon value of all flows subsequent to the horizon and strictly attributable to the present plant. In the present calculations it may be ignored since it is an initial condition — that is, a constant (however it is determined) — and will not be affected by events subsequent to the present. Hence we need to concern ourselves only with the optimum horizon value of the earlier model which we shall denote by A, that is,

$$A = \text{Maximum } [\sum_j \hat{a}_j x_j + v_T - w_T], \tag{9.25}$$

subject to constraint set (8.1).

We denote the number of shares of stock outstanding at time $t = 0$ by S, and the number of shares issued in year t by s_t. We shall make one other strong assumption, namely, that the only concern is the per-share value at the horizon and not this value at any time before the horizon. Thus we shall only seek to find that number of shares to be issued and the date of their issuance which maximizes the per-share value of the firm at the horizon. We also assume the yield to the firm

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[32]This assertion disregards the element of dilution of control which is also involved in equity financing. As mentioned before, this and the earlier models are approaches to investment decisions under uncertainty. Equity financing especially, but also rising supply curves for borrowed funds, are a result of the presence of risk, due to uncertainty attached to the outcomes of investments. The stochastic treatment of outcomes, required in a full uncertainty model, is beyond the scope of the present study.

[33]To the extent the firm possesses opportunities in the future subsequent to the horizon, the functional of model (8.1) understates the value of the firm. Insofar as these future opportunities do not require outlays in the interval up to the horizon, they do not enter into the decision problem here.

of a share of stock issued in year t to be known, and we denote this quantity by p_t.[34]

The procedure is divided into two phases. In the first we solve the appropriate model without stock financing, determine the value of A in (9.25) above, and also the value of A/S, the per-share value of the firm without additional equity capital. In the second phase the constraints of this model are modified by adding the proceeds of stock financing.[35] Further, we add a constraint to ensure that stock financing, if undertaken, will actually be profitable to the present owners.[36] The model for the second phase then is

$$\begin{aligned}
&\text{Maximize} && \sum_j \hat{a}_j x_j + v_T - w_T \\
&\text{Subject to (a)} && \sum_j a_{1j} x_j + v_1 - w_1 - p_1 s_1 \leq D_1 \\
&\text{(b)} && \sum_j a_{tj} x_j - (1 + r) v_{t-1} + v_t \\
& && \qquad + (1 + r) w_{t-1} - w_t - p_t s_t \leq D_t, \\
& && \qquad t = 2, \ldots, T \\
&\text{(c)} && 0 \leq x_j \leq 1 \\
&\text{(d)} && v_t, w_t \geq 0,[37]
\end{aligned} \tag{9.26}$$

plus one additional constraint whose nature will now be developed both verbally and analytically.

Profitability of stock financing for present owners is based on the "before" and "after" per-share values of the firm. Denoting the optimal

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[34]This assertion by itself, admittedly, has the effect of merely converting the borrowing rate of earlier models into a price. This is not our intention. Instead we suppose that conditions can be written down which determine the price p_t as a function of other variables of the model, making this a less trivial exercise. However, stating the form of these functions may require a volume in itself, and no attempt in this direction will be undertaken here. (From a technical point of view, fixed costs of stock flotation may be handled by integer programming methods in ways similar to those of the preceding model.)

[35]It may be desirable to lag the proceeds of the stock sale.

[36]This ignores the possibility that some future owners may find subsequent stock financing unprofitable. This situation, which has not been fully explored, could lead to an advance in timing of stock issues so as to increase the earlier amounts, and to combining this change with lending until the time the additional funds are needed.

[37]The borrowing constraints, if any, would also be added here.

value of the functional in (9.26) by A', we require that

$$\frac{A'}{S + \sum_t s_t} \geq \frac{A}{S}, \tag{9.27}$$

or, that

$$A' - \left(\frac{A}{S}\right)\sum_t s_t \geq A. \tag{9.28}$$

While A' contains variables of the problem, both A and S are parameters in this second phase. Hence (9.28) is a valid linear restriction. The final step requires finding the *maximum* per-share value. To locate this maximum, assuming a solution satisfying restriction (9.28) exists, we must utilize parametric linear programming.

We seek the value of $\sum_t s_t$ and the separate values of the s_t which maximize the left side of inequality (9.27). To accomplish this we add the constraint

$$\sum_t s_t \leq \theta \tag{9.29}$$

and let the value of θ vary over the range $0 \leq \theta \leq \infty$ by means of parametric right side, discussed in Section 7.3 above. For every value of θ for which the variables of the solution change[38] we compute the ratio $A'/(S + \sum_t s_t)$ and choose that set of s_t corresponding to the highest ratio.[39] This yields the optimum amount of stock financing together with the optimum timing, as well as the choices of the investment alternatives and the other financial transactions.

9.6. Conclusion

In the present chapter we have shown how the programming approach may be used to find solutions to the problem of capital allocation in the presence of various forms of financial market imperfection. Our point in doing so has been twofold. First, we have demonstrated that the common criteria for investment decisions are not appropriate tools for

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[38]That is, for every change of basis.

[39]The procedure is actually somewhat simpler. Using the theorem that the optimal values of the functional are a convex function of the parameter θ, we know that once the ratio $A'/(S + \Sigma s_t)$ becomes smaller after increasing, for increasing values of θ, we have reached and passed the maximum, and so end the computations.

choosing among investments when there are limits to borrowing at a given rate of interest. In fact, we may assert that these criteria are not tools of capital *budgeting* at all since they cannot be used to decide among investments within budget limitations of any kind. However, much of this we have already seen in the earlier Lorie-Savage models.

The main emphasis in this chapter has been to focus on some of the effects commonly attributed to uncertainty — rising supply curve for borrowed funds, absolute borrowing limits, nonrenewable short-term loans, and equity financing — and to show how the methods of linear and integer programming may be used to determine the optimal investment program. In the appendix following this chapter we will present numerical examples to illustrate some of these models.

APPENDIX 9A

Numerical Examples of Investment Decisions by the Horizon Models under Perfect and Imperfect Capital Markets, and with and without Interdependencies between Projects

The following five numerical examples illustrate some of the models of Chapters 8 and 9 and demonstrate how the common criteria for investment decisions — present value and internal rate of return — may mislead in the presence of interdependencies among investment projects or in less than perfect capital markets.[1] The data for these problems consist of the project flows

[1]The problems in this appendix were designed for a number of purposes, and are somewhat more complex than they need to be to illustrate these models. The calculations made use of the SCROL LP Code developed by CEIR, Inc. for the IBM 704 computer. The versatility of this code made it possible to set up the problem in its dual form, then to modify it by the addition of columns (instead of adding rows to the primal problem) and thereby to solve the entire series of problems without starting any but the first from the beginning. The results, however, will be presented in terms of the primal problem.

TABLE 9A.1.

CASH FLOWS ASSOCIATED WITH THIRTY HYPOTHETICAL INVESTMENT PROJECTS[a]

Project Number	Flow in Year																								
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	−100	20	20	20	19	19	18	16	14	11	6	−8														
2	−100	20	18	18	18	18	14	14	14	14	14	10	10	10	10	10	6	6	6	6	6					
3	−100	15	15	15	15	15	13	13	13	13	13	11	11	11	11	11	9	9	9	9	9					
4	−100	20	6	11	7	16	5	14	18	3	20	2	22	8	10	18	6	9	14	24						
5	−100	−60	−60	80	74	66	56	44	30	14																
6	−200	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
7							−80	20	20	20	19	17	14	10	6	2										
8									−60	−30	−10	45	34	25	16	12	8	−20	21	16	12	9	7	5	3	
9			−120	25	25	30	35	30	25	20	15	10	5													
10		−100	18	17	15	12	8	−10	18	17	15	12	8	−10	18	17	15	12	8							
11	−150	20	20	20	20	20	20	20	20	20	20	20														
12	−100	20	18	16	14	12	10	4	−20	20	18	16	14	12	10	4										
13	−150	−75	−75	60	60	55	50	44	38	36	35	34	33	30	25	17	9									
14	−50	−100	−175	50	55	60	65	60	50	40	30	20	10	−25	50	41	35	25	15	5						
15	−100	−150	−100	10	20	30	40	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
16								−95	−60	47	42	37	31	24	18	13	9	6	4	3						
17										−175	50	45	35	25	10	−60	45	35	25	10						
18	−250	45	45	40	30	25	20	15	10	−40	40	32	25	19	14	10	7	5								
19	−75	−75	−40	40	40	40	35	35	30	25	15	5														
20	−180	20	12	16	13	11	19	17	12	15	19	13	14	17	20	14	11	15	17	12						
21												−80	18	16	14	12	10	4	16	14	10	6				
22		−85	20	20	16	15	13	10	7	3																
23		−270	−100	125	115	105	80	60	35	25	15	10														
24										−40	15	13	9	7	5	2										
25								−50	10	10	9	7	4	−14	9	9	8	6	3	−16	8	8	4			
26		−200	60	40	30	15	−25	−25	50	40	30	20	10													
27											−70	15	13	11	10	9	7	6	4	3	2					
28		−355	60	70	80	70	55	40	25	15	5															
29	−275	40	45	45	40	35	30	25	20	15	−75	35	30	25	20	15	10	5								
30	−140	20	20	18	16	14	11	8	−25	18	18	16	13	10	6	−25	16	16	14	11	8	5	2			

[a] $(-a_{ij})$

TABLE 9A.2. HORIZON VALUES OF INVESTMENT PROJECTS[a]

Project No.	6	8	15	21	25	30
Horizon Value (10%)	94.77	20.47	227.45	5.45	10.50	6.20
Horizon Value (5%)	108.24	21.71	259.77	5.71	11.25	6.58

[a]($\hat{a}_j$). $T = 21$. These values are computed at both 10% and 5% discount rates. (Projects having no flows past $t = 21$ have zero horizon values.) See expression (8.10), p. 144.

TABLE 9A.3. HYPOTHETICAL CASH THROW-OFF FOR TWENTY-ONE YEARS[a]

Year	1	2	3	4	5	6	7	8	9	10	11
Amount	$400	360	320	280	240	200	160	120	80	40	0
Year	12	13	14	15	16	17	18	19	20	21	
Amount	−40	−80	−120	−160	−200	−240	−280	−320	−360	−400	

[a](D_t)

TABLE 9A.4. INTERNAL RATES OF RETURN AND RANKS FOR THIRTY INVESTMENT PROJECTS[a]

Project No.	1	2	3	4	5	6	7	8
Rate (%)	11.03	13.94	11.90	10.02	12.26	11.75	13.84	12.57
Rank	9	2	7	13	5	8	3	4
Project No.	9	10	11	12	13	14	15	16
Rate (%)	15.66	9.07	7.00	8.55	8.76	9.22	10.86	10.24
Rank	1	16	19	18	17	15	10	11
Project No.	17	18	19	20	21	22	23	24
Rate (%)	5.81	5.76	6.75	5.19	9.35	6.11	11.94	10.19
Rank	22	23	20	24	14	21	6	12
Project No.	25	26	27	28	29	30		
Rate (%)	4.52	4.25	3.50	4.71	4.64	4.93		
Rank	28	29	30	26	27	25		

[a]These rates are computed by using the horizon values at 10% of Table 9A.2.

($-a_{tj}$) presented in Table 9A.1, the horizon values of flows past the horizon ($\hat{a}_j$) given in Table 9A.2, and the projected cash throw-off (D_t) shown in Table 9A.3. The horizon for these problems is $T = 21$ years. Some of the thirty projects, whose internal rates of return are presented in Table 9A.4, have flows commencing several periods after the first, and some have outflows considerably later than their initial outlays. These represent either demolition costs, if they occur at the end, or they may indicate expenditures for renovating such as occur in rebuilding a blast furnace. As will be seen from the table, the cash flows are not, in general, assumed to be level. Also, the cash throw-off (attributable only to assets existing currently, by definition) is assumed to become negative from the twelfth year on.

Problem 1.

This problem is a straightforward application of model (8.1) to these data, that is, it calls for the selection of the optimal set of investment proposals with the availability of unlimited lending and borrowing at an interest rate of 10 per cent. The optimal solution to this problem is presented in Table 9A.5. The projects selected, numbers 1 through 9, 15, 16, 23, and 24, all have positive present values when computed at the market interest rate of 10 per cent as the discussion in Section 8.2 stated must happen. These present values for all projects are given by the dual variables: μ_j^* for the accepted projects, and

TABLE 9A.5. SOLUTION TO PROBLEM 1

Accepted Projects			
Project	x^*	μ^*	Rank
1	1.0	$ 24.14	10
2	1.0	142.57	6
3	1.0	78.75	7
4	1.0	0.05	13
5	1.0	145.44	4
6	1.0	181.15	1
7	1.0	37.79	8
8	1.0	36.11	9
9	1.0	144.57	5
15	1.0	179.48	2
16	1.0	9.35	11
23	1.0	146.15	3
24	1.0	0.45	12

Rejected Projects		
Project	γ^*	Rank
10	$ 33.94	15
11	135.21	23
12	46.98	18
13	127.35	22
14	79.38	21
17	64.95	20
18	339.38	27
19	162.67	24
20	353.17	29
21	4.57	14
22	56.97	19
25	36.89	17
26	250.69	25
27	36.31	16
28	342.56	28
29	410.07	30
30	252.97	26

Yr.	Amount Borrowed (10%)	Amount Lent
1	$400.00	
2	460.00	
3	482.00	
4		$ 78.80
5		644.68
6		1233.15
7		1722.46
8		2215.71
9		2651.28
10		3138.41
11		3686.25
12		4246.87
13		4812.56
14		5353.82
15		5890.20
16		6432.22
17		6958.44
18		7469.28
19		8035.21
20		8615.73
21		9183.31

Program Value = $9526.00.

$-\gamma_j^*$ for the rejected ones. Also as expected, the rankings given by the dual variables, in this instance the present values, do not coincide with the rankings produced by the internal rates of return, although, of course, these two criteria yield the same list of acceptable projects. Lending or borrowing takes place in every year; hence, no fractional projects appear.

Problem 2.

This problem is an application of model (9.1) in which absolute limits are placed on the amount of borrowing — in this instance \$200 in each year. As

TABLE 9A.6. SOLUTION TO PROBLEM 2

Accepted Projects			
Project	x^*	μ^*	Rank
2	1.0	\$ 93.86	2
3	1.0	27.32	8
5	1.0	53.42	4
6	1.0	75.59	3
7	1.0	37.79	6
8	1.0	36.11	7
9	1.0	144.57	1
15	1.0	38.75	5
16	1.0	9.35	9
23	0.7	0.00	11
24	1.0	0.45	10

Rejected Projects		
Project	γ^*	Rank
1	\$ 24.58	13
4	48.67	16
10	88.07	18
11	213.70	22
12	95.70	19
13	257.26	24
14	163.28	21
17	64.95	17
18	463.88	28
19	247.92	23
20	449.52	27
21	4.57	12
22	102.98	20
25	36.89	15
26	358.95	26
27	36.31	14
28	534.71	29
29	552.16	30
30	325.50	25

Yr.	Amount Borrowed (10%)	Amount Lent
1	\$200.00	
2	200.00	
3	192.37	
4		\$ 329.36
5		860.22
6		1404.13
7		1863.84
8		2323.44
9		2727.42
10		3200.75
11		3724.38
12		4291.86
13		4840.04
14		5376.05
15		5904.65
16		6430.12
17		6950.13
18		7451.14
19		8001.25
20		8554.38
21		9115.82

Program Value = \$9458.51.

the solution given in Table 9A.6 shows, this borrowing limit is operative in the second year only, and the effective rate of interest for that year, $\rho_4^*/\rho_3^* - 1$, rises to 19.7 per cent, which is greater than the market interest rate of 10 per cent. Although the amount of borrowing in year one is only \$200, from the value of ρ_1^* we learn that no greater amount would have been borrowed in the absence of this limit in the first year. It would also have been possible to obtain this information by the alternate route of adding up the outlays and inflows of the accepted projects and the first year cash throw-off.

The set of accepted projects is now the same as for the preceding problem except that projects 1 and 4 are eliminated and that project 23 becomes fractional. The last-named is an attractive project, having the third highest present value at the 10 per cent market rate of interest. However, it has a very high outflow in the second year, reducing its attractiveness in the presence of borrowing limits. By contrast, neither project 1 nor project 4 has an outflow in the second year, but each generates an inflow instead. Their rejection may be traced to the fact that adopting them would require foregoing other projects. Both projects 1 and 4 require outlays in the first year, and their acceptance would make the borrowing limit effective. While project 4 was barely inframarginal before, project 1 ranked tenth by the present value criterion as opposed to a rank of 13 for the lowest accepted project in Problem 1. However, the projects which ranked eleventh and twelfth do not have any flows until later years at which times lending takes place and borrowing limits are not operative. Similarly three projects which are ranked below project 1 by the internal rate of return are still retained in the optimal solution. These are projects 15, 16 and 24. As this problem demonstrates, timing of the flows may become crucial in project selection as soon as borrowing limits are imposed.

The dual variables μ_j^* and γ_j^* are now the present values computed at the rate of 10 per cent in all years but the second, in which the rate of 19.7 per cent is used. We may note, in passing, that the ranks of the previously accepted projects also change when ranking is accomplished by use of these dual variables. For example, project 23, which ranked third before, now ranks last among the accepted projects.

Problem 3.

In this problem, model (9.14) is applied to the same set of data. A sloping supply curve for funds is substituted for limited borrowing, with marginal interest rates set at 10 per cent, 12 per cent and 14 per cent. The maximum amounts that may be borrowed at each of the lower two rates is \$200 in each year. The effect of these financing opportunities, as shown in Table 9A.7, is to make project 23 integral again, but not to restore projects 1 and 4 to the optimal set. Borrowing at 12 per cent takes place in the second and third years, but no borrowing takes place at 14 per cent in any year. The effective marginal rate of interest beomes 12 per cent for years 2 and 3, and 10.9 per cent in year 1. That is, while only \$200 is borrowed in the first year, more would have been borrowed if

TABLE 9A.7. SOLUTION TO PROBLEM 3

Accepted Projects			
Project	x^*	μ^*	Rank
2	1.0	$118.23	3
3	1.0	52.98	7
5	1.0	95.26	5
6	1.0	127.98	2
7	1.0	37.79	8
8	1.0	36.11	9
9	1.0	132.44	1
15	1.0	105.06	4
16	1.0	9.35	10
23	1.0	75.45	6
24	1.0	0.45	11

Rejected Projects		
Project	γ^*	Rank
1	$ 0.00	12
4	25.51	14
10	54.56	17
11	174.68	22
12	71.32	19
13	197.74	23
14	134.84	21
17	64.95	18
18	401.36	27
19	206.53	24
20	402.64	28
21	4.57	13
22	74.02	20
25	36.89	16
26	289.51	26
27	36.31	15
28	416.16	29
29	480.83	30
30	289.37	25

Yr.	Amount Borrowed (10%)	Amount Borrowed (12%)	Amount Lent
1	$200.00		
2	200.00	$ 80.00	
3	200.00	111.60	
4			$ 233.01
5			788.31
6			1356.14
7			1834.75
8			2309.23
9			2722.15
10			3202.37
11			3730.60
12			4301.66
13			4850.83
14			5387.91
15			5917.70
16			6444.48
17			6965.92
18			7468.51
19			8020.37
20			8575.40
21			9138.94

Program Value = $9481.63.

it had been available at a marginal rate less than 10.9 per cent, even though greater than 10 per cent.

The dual variables μ_j^* and γ_j^* are again present values of the projects computed at the marginal *effective* interest rates. According to these, the rank of project 23, which was thirteenth in Problem 2 now becomes sixth. Its heavy drain on cash in the second year is no longer an insurmountable obstacle, even though borrowing must take place at 12 per cent. The program value is increased over that of the preceding problem by $23.12. Project 1, we may note, receives a zero present value in this solution, indicating that at least part of it

might be included without loss, if this were desirable from other points of view. A minor side calculation will show that it can be included in its entirety without reducing the optimal program value, although the linear programming solution does not indicate this directly.

Problem 4.

In this problem the model of Problem 1 is augmented by a set of four restrictions expressing relationships of competitiveness or complementarity between some of the projects. No borrowing limits are imposed in this problem. The added restrictions are

$$\begin{aligned} &\text{(a)} \quad x_2 + x_3 + x_{11} \leq 1 \\ &\text{(b)} \quad x_{13} + x_{14} \leq 1 \\ &\text{(c)} \quad x_{15} - x_{30} \leq 0 \\ &\text{(d)} \quad x_7 + x_8 - x_{12} \leq 0. \end{aligned} \tag{9A.1}$$

The solution to this problem is presented in Table 9A.8. From among the set of mutually exclusive projects of inequality (9A.1), project 2 is accepted. This constraint introduces a new variable into the dual problem, which we shall denote by ν. Interpretation of the dual variables for the projects involved in the added constraints is altered. The value of the accepted project from the mutually exclusive set is the amount which would be foregone if this project were replaced by the next best alternative, in this instance the value of project 3. The new dual constraint results in the relation

$$\hat{a}_2 - \sum_t \rho_t^* a_{t,2} = \mu_2^* + \nu^* \tag{9A.2}$$

for the accepted project. We know from the solution to Problem 1 that the left side of this equation is \$142.57, which is the sum of $\mu_2^* = \$63.83$ and $\nu^* = \$78.75$. For the rejected projects j,

$$\sum_t \rho_t^* a_{tj} - \hat{a}_j = \gamma_j^* - \nu^*. \tag{9A.3}$$

Again, from Table 9A.5 we know that the left side for project 3 is $-\$78.75$ which is $\gamma_3^* = 0$ less $\nu^* = \$78.75$. For project 11 these quantities are \$135.21 = \$213.96 − \$78.75. In other words, while project 3 had been accepted in Problem 1 but project 11 had not, the imposition of the limit of one from this set of three projects results in the choice of project 2, and the value of the dual variables of the rejected ones is the opportunity cost of their acceptance. For purposes of ranking projects only the accepted project has been included, and its value is given by $\mu_2^* + \nu^*$.

Neither project in restriction (9A.1b) is accepted, as their present values given in Table 9A.5 indicate that they should not be. Hence the dual variables γ_{13}^* and γ_{14}^* remain unchanged from Table 9A.5 in Table 9A.8. For purposes of ranking, only γ_{14}^*, the numerically smaller of these values, is used.

TABLE 9A.8. SOLUTION TO PROBLEM 4

Accepted Projects			
Project	x^*	μ^*	Rank
1	1.0	$ 24.14	6
2	1.0	[a]	5
4	1.0	0.05	9
5	1.0	145.44	3
6	1.0	181.15	1
9	1.0	144.57	4
16	1.0	9.35	7
23	1.0	146.15	2
24	1.0	0.45	8

Rejected Projects		
Project	γ^*	Rank
3	[a]	[b]
7	[a]	[b]
8	[a]	[b]
10	$ 33.94	12
11	[a]	[b]
12	[a]	11
13	[a]	[b]
14	[a]	18
15	[a]	[b]
17	64.95	16
18	339.38	21
19	162.67	19
20	353.17	23
21	4.57	10
22	56.97	15
25	36.89	14
26	250.69	20
27	36.31	13
28	342.56	22
29	410.07	24
30	[a]	17

Yr.	Amount Borrowed (10%)	Amount Lent
1	$200.00	
2	105.00	
3	6.50	
4		$ 576.85
5		1157.53
6		1752.29
7		2320.52
8		2780.57
9		3239.63
10		3722.59
11		4246.85
12		4730.53
13		5225.59
14		5702.14
15		6180.36
16		6666.39
17		7139.04
18		7618.94
19		8109.83
20		8612.81
21		9099.09

Program Value = $9193.86.

[a]See reference to value in text of Problem 4.
[b]See reference to ranking in text of Problem 4.

Restriction (9A.1c) makes acceptance of project 15 dependent on adoption of project 30. Since their combined present value, as given in Table 9.5, is less than zero, as is the present value of the independent project alone, neither is accepted in this solution. For the independent member, the dual restriction becomes

$$\sum_t \rho_t^* a_{t,30} - \hat{a}_{30} = \gamma_{30}^* + \nu^* \tag{9A.4}$$

while for the dependent one it becomes

$$\sum_t \rho_t^* a_{t,15} - \hat{a}_{15} = \gamma_{15}^* - \nu^*. \tag{9A.5}$$

For the independent member the left side is again the negative of the value in Table 9A.5, which here is divided into the component of loss (γ_{30}^*) attributable to forced acceptance of the project (which is offset by the value of the dependent member which such acceptance makes possible) and the value of the dependent member itself, which is the value of the dual ν^* for this restriction. Numerically, \$252.97 = \$73.49 + \$179.48. For the dependent member, the variable x_{15} appears in the basis set, but at zero level. Relation (9A.5) then is the negative of the present value of this project, as given in Table 9A.5, on both sides of the equal sign. This amount is \$179.48 = ν^* since $\gamma_{15}^* = 0$. Only the independent member of this restriction is ranked with the other projects, the rank itself being given by the value of γ_{30}^*. In general, the value for ranking should be the higher of the present value of the independent member and the value of the "compound project" which the independent project makes possible.

Finally, in restriction (9A.1d) two projects are considered mutually exclusive and both are dependent on a third project. None of the projects is accepted. The combined present values of either projects 7 and 12 or 8 and 12 are negative. (Both projects 7 and 8 were accepted in Problem 1 although project 12 was not.)[2] Equations of the form of (9A.4) and (9A.5) apply to the independent and dependent projects, respectively. For the former we have here

$$\$46.98 = \$9.19 + 37.79 = \gamma_{12}^* + \nu^*$$

while for the latter we have

(a) for project 7: $-\$37.79 = 0 - \$37.79 = \gamma_7^* - \nu^*$

and

(b) for project 8: $-\$36.11 = \$1.68 - \$37.79 = \gamma_8^* - \nu^*$.

For purposes of ranking projects the value of γ_{12}^* here seems to be the most appropriate. This is the smallest loss which would result if the independent member were required to be accepted.

Problem 5.

In this problem unlimited amounts for lending and borrowing are also made available, but at different rates of interest. The contingency relations between projects have been dropped. The "borrowing rate" remains at 10 per cent, but the "lending rate" becomes 5 per cent. One effect of this change is to require a decision with respect to the discount rate for computing the horizon values of flows subsequent to the horizon. Values for both 5 per cent and 10

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[2]Present values of these projects are again found in Table 9A.5.

TABLE 9A.9. SOLUTION TO PROBLEM 5

Accepted Projects			
Project	x^*	μ^*	Rank
1	1.0	$ 21.79	16
2	1.0	117.12	3
3	1.0	99.88	6
4	1.0	67.52	10
5	1.0	95.88	7
6	1.0	295.38	2
7	1.0	52.11	12
8	1.0	76.86	9
9	1.0	102.68	5
10	1.0	33.39	14
12	1.0	8.93	17
13	1.0	43.09	13
14	1.0	92.40	8
15	1.0	580.06	1
16	1.0	65.04	11
17	1.0	8.93	18
21	1.0	23.91	15
23	1.0	114.28	4
24	1.0	8.59	19

Rejected Projects		
Project	γ^*	Rank
11	$ 37.34	23
18	112.63	28
19	50.70	24
20	96.39	27
22	23.00	22
25	1.52	20
26	86.09	26
27	6.17	21
28	146.94	29
29	153.40	30
30	75.71	25

Yr.	Amount Borrowed (10%)	Amount Lent
1	$ 700.00	
2	1045.00	
3	1339.50	
4	721.45	
5	91.59	
6		$ 562.25
7		1089.36
8		1562.83
9		1940.97
10		2198.02
11		2689.92
12		3063.41
13		3475.58
14		3757.36
15		4073.23
16		4260.89
17		4470.94
18		4585.48
19		4697.76
20		4738.64
21		4691.58

Program Value = $5087.01.

per cent discount rates are given in Table 9A.2. The effect of these different discount rates is nil for the present problem, and it is apt to be small in general. However, it is always possible to utilize the appropriate one, which depends on whether "lending" or "borrowing" will take place in the post-horizon period. In these computations the lending rate of 5 per cent was used for discounting these horizon values.

The solution to the problem is given in Table 9A.9. The number of projects accepted in this solution is increased considerably over that in Problem 1, and borrowing now takes place in the first five years. The accepted projects have

positive present values (μ^*) computed at 10 per cent for the first five years, and at 5 per cent thereafter. This implies that no simple ranking scheme carried out in advance of the decision to accept or reject the individual projects is possible. It depends on whether lending or borrowing takes place.[3]

For example, the ranks of projects 6 and 15, which ranked first and second respectively, in Problem 1, are now reversed. Similarly, the internal rate of return criterion leads to contradictory results in this situation. Project 11, with an internal rate of return of 7.00 per cent is rejected, while project 17, with an internal rate of return of 5.81 per cent, is accepted. The former has heavy outflows in early years which are discounted at 10 per cent while the latter does not begin until later years when the 5 per cent rate applies.[4]

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[3]If neither lending nor borrowing takes place in any year, the discount rate for that year would lie in the closed interval of 5–10 per cent, which would be the "marginal productive opportunity rate" in the language of Hirshleifer (*op. cit.*, p. 335).

[4]Although the amount of lending and borrowing is unaffected, the values of the dual variables of the projects, μ^* and γ^*, will be altered slightly if the horizon values computed at 10 per cent are used.

CHAPTER 10

Conclusions

10.1. Summary

In the course of this study we have examined the basis for and the consequences of the use of capital budgets in investment decisions. We have shown that Joel Dean's decision procedure of ranking investment proposals by their rates of return and selecting all above some cut-off rate or interest often fails to be operational without a wide range of prior calculations. Furthermore, as Lorie and Savage have demonstrated, it is inadequate in the presence of multiple budgets and nonindependent investment alternatives.

A linear programming formulation was given for the Lorie-Savage problem and for a number of related problems involving multiple budget ceilings and complex interrelationships between investments. It was also shown that an exact solution to these problems requires the application of integer programming. This formulation was used to demonstrate the failure of the Lorie-Savage method of allocating fixed dollar budgets among investment alternatives. The reasons for the breakdown of their procedure — which could be interpreted as an attempt to solve the dual problem to the linear programming formulation — were analyzed and shown to be related to the possible nonexistence of prices consistent with efficient allocation under decentralization of investment decisions in the presence of indivisibilities.

We have also examined in detail the properties of the recomputed dual prices proposed by Gomory and Baumol for the integer programming problem and have shown how alternative dual evaluators may be generated by use of linear programming in combination with the integer

programming solution. We have demonstrated the properties of these alternative dual variables, including some desirable properties not shared by the Gomory and Baumol recomputed prices.

The Lorie-Savage models were extended to situations of multiple budgets such as arise in practice when personnel or materials constitute binding limitations on capital expenditure programs. In this context it was shown that nonlinear relationships expressing opportunities for substitution of capital outlays for other scarce resources could be handled within the same framework.

Finally, the capital budgeting problem was reformulated along the lines of Charnes, Cooper and Miller, using linear programming as a technique for integrating the physical investment decisions with the financing decisions. This framework was exploited to exhibit the traditional criteria for choosing among independent investment proposals under certainty and perfect capital markets, and also to show the failure of these criteria under various other forms of capital market imperfections and interrelationships between investment projects. Two models involving such imperfections were presented in which long-term borrowing at bank rates is precluded, although unlimited borrowing through the use of bonds or equity financing is made available. These models are not only of some interest in their own right but may be considered a first step toward the incorporation of uncertainty since they do allow for some of the visible effects of at least the lender's uncertainty with respect to the repayment of loans.

An extended set of numerical examples was provided to exhibit a number of these results.

10.2. Some Implications

We have shown the mathematical programming formulation of capital budgeting to be capable of setting forth the implications of various criteria for investment decisions. This framework also provides an operational and computationally feasible method of solving the vast combinatorial problem implied in the construction of Fisher's investment opportunity line (often called the transformation curve[1]), and also of resolving the controversy over which rate — the lending or the borrowing rate — is the appropriate rate of interest to use in discounting streams of costs and revenues in applying the present value criterion.[2] It demon-

[1]Fisher, *op. cit.*, pp. 264 ff. See also R. G. D. Allen, *op. cit.*

[2]See Hirshleifer, "On the Theory of Optimal Investment Decisions," *op. cit.*

strates, in addition, that discounting itself — and especially discounting at assumed *market* rates of interest — is an artificial procedure which, although designed to simplify the decision process, obscures much of what needs to be taken into consideration, and which, in any case, may not be required, or may even be misleading.

One important implication of our analysis of optimal allocation under multi-period budget constraints is especially relevant to governmental decisions. The current practice of setting first year spending ceilings on classes of expenditures, such as water resource and electric power projects, foreign aid programs and weapon systems development, while delaying statement of later constraints, is almost sure to lead to inefficient allocation of limited funds. Although projects of these types require outlays in many years and the alternative of project postponement usually exists, the decision makers are required to guess what future budgets will be in deciding what projects to carry out currently. Explicit statement of future limits, to the extent feasible, would make possible some improvement in these decisions, especially in regard to the scale and timing of these projects.[3]

10.3 Some Directions for Future Research

A number of extensions of the work carried out may be suggested. The first of these is a more explicit treatment of uncertainty, not merely as was done here, in terms of conditions imposed by the lender of funds, but also to include the outcome of investments as stochastic variables. Recently some developments in linear programming under uncertainty have appeared which may make such extensions possible.[4]

Related to the "frontal attack" on the problem of uncertainty is the exploration of the dynamic properties of an iterative decision procedure

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[3]On this point, see especially Thomas Marschak, "Capital Budgeting and Pricing in the French Nationalized Industries," *Journal of Business*, XXXIII, No. 2 (April, 1960), pp. 133–56.

[4]See A. Charnes and W. W. Cooper, "Chance-Constrained Programming," *Management Science*, VI, No. 1 (October, 1959), pp. 73–79; A. Charnes and W. W. Cooper, "Chance Constraints and Normal Deviates," *Journal of the American Statistical Association*, LVII, No. 257 (March, 1962), pp. 134–48. G. Dantzig, "Linear Programming Under Uncertainty," *Management Science*, I, Nos. 3 and 4 (April and July, 1955), pp. 197–206; S. E. Elmaghraby, "An Approach to Linear Programming Under Uncertainty," *Operations Research*, VII, No. 2 (March-April, 1959), pp. 208-16; and B. Naslund and A. Whinston, "A Model of Multi-Period Investment Under Uncertainty," *Management Science*, VIII, No. 2 (January, 1962), pp. 184–200.

in which models of the type formulated are solved periodically, each time with more recent and hence better estimates of future flows resulting from current investments and more complete statement of future investment alternatives.[5] In this regard, it is also necessary that much more attention be given to financial questions, especially those related to the cost of capital and alternative sources of financing investments.[6] Recent advances in integer programming and mixed integer programming suggest formal solutions to problems of fixed financing charges, although practical applications of these methods are not yet at hand.

Finally, an area that has been by-passed almost entirely in the present study is the organizational and administrative side of capital budgeting. The achievement of economically rational investment decisions almost certainly requires more attention to these kinds of considerations along with suitable computational procedures. Our discussion of the dual variables in integer programming has given only an initial hint of the problems that may be encountered here since we (along with Gomory, Baumol, *et al.*) have not really treated the considerations that are likely to be encountered in real firms. Something more than formal recognition must be given to organizational structure if we are to deal with the real problems of decentralization in decision-making for capital budgeting in the face of constraints that are imposed by the structures of firms.[7]

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[5]This is the approach taken in the articles on "Chance-Constrained Programming" by Charnes and Cooper, cited above.

[6]See, for example, F. Modigliani and M. H. Miller, "The Cost of Capital, Corporation Finance and the Theory of Investment," *op. cit.;* D. Durand, "The Cost of Debt and Equity Funds for Business," *Conference on Research in Business Finance* (New York: National Bureau of Economic Research, 1952), pp. 215–47; M. J. Gordon, "Security and a Financial Theory of Investment," *The Quarterly Journal of Economics,* LXXIV, No. 3 (August, 1960), pp. 472–92.

[7]See, for example, R. M. Cyert and J. G. March, "Organizational Factors in the Theory of Oligopoly," *Quarterly Journal of Economics,* LXX, No. 1 (February, 1956), pp. 44–64.

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